**FIRST EVALUATION OF ALGEBRA II**
**(September 28th, 2017)**

1. Given the cone

$$\mathbf{A}_\pi = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 & 1 & 0 & 0 & 2 \\ -1 & 1 & 0 & 1 & 0 & 1 & 2 & 0 \end{pmatrix}_\pi . \tag{1}$$

Answer the following questions:

*a*) Obtain the dual cone $\mathbf{A}_\pi^p$ (8 points).

*b*) Obtain the edges of the cone $\mathbf{A}_\pi$ (4 points).

*c*) Obtain the main facets of the cone $\mathbf{A}_\pi$ (4 points).

*d*) Obtain the edges of the cone $\mathbf{A}_\pi^p$ (4 points).

*e*) Obtain the dual cone of the cone (4 points)

$$\mathbf{B}_\pi = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 1 & 2 \end{pmatrix}_\pi .$$

*f*) Investigate if the vectors $(1,0,0,0,0)$, $(0,1,0,0,0)$, $(0,0,1,0,0)$, $(0,0,0,1,0)$ y $(0,0,0,0,1)$ belong to the cone $\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & -1 & 1 \\ -1 & 1 & 0 & 1 & 0 \end{pmatrix}_\pi$. (4 points).

*g*) Determine the conditions under which the following system is compatible (8 points):

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ -1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \\ a \\ -a \end{pmatrix} \tag{2}$$

$$x_1, x_2, x_3, x_4 \geq 0. \tag{3}$$

*h*) Obtain the solution of the system: (8 points):

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \leq \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \tag{4}$$

*i*) Obtain the solution of the system containing the first three equations of the system (4) (6 points).

**STUDENT'S FIRST AND LAST NAME .....**

1. **Solution of Problem 2 (50 points):**

<table>
<tr><td colspan="4">(a) (8 points)</td></tr>
<tr><td colspan="4"></td></tr>
<tr><td>(b) (4 points)</td><td>(c) (4 points)</td><td>(d) (4 points)</td><td>(e) (4 points)</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td>(f) (4 points)</td><td colspan="3">(g) (8 points)</td></tr>
<tr><td></td><td colspan="3"></td></tr>
<tr><td colspan="2">(h) (8 points)</td><td colspan="2">(i) (6 points)</td></tr>
<tr><td colspan="2"></td><td colspan="2"></td></tr>
</table>

**SOLUTIONS TO THE FIRST EVALUATION OF ALGEBRA II:**
(September, 28th, 2017)

1. SOLUTION TO PROBLEM 2.

*a*) **Solution:** First, we apply the $\Gamma$, algorithm and obtain the tables:

**Iteration 1**

| $\mathbf{a}_1$ | $\mathbf{v}_1^1$ | $\mathbf{v}_2^1$ | $\mathbf{v}_3^1$ | $\mathbf{v}_4^1$ | $\mathbf{v}_5^1$ |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | **0** |
| 0 | 0 | 1 | 0 | 0 | **0** |
| 0 | 0 | 0 | 1 | 0 | **0** |
| 0 | 0 | 0 | 0 | 1 | **0** |
| -1 | 0 | 0 | 0 | 0 | **1** |
| $\mathbf{t}^1$ | 0 | 0 | 0 | 0 | **-1** |

**Iteration 2**

| $\mathbf{a}_2$ | $\mathbf{v}_1^2$ | $\mathbf{v}_2^2$ | $\mathbf{v}_3^2$ | $\mathbf{v}_4^2$ | $\mathbf{w}_1^2$ |
|---|---|---|---|---|---|
| 1 | **1** | 0 | 0 | 0 | 0 |
| 1 | **0** | 1 | 0 | 0 | 0 |
| 1 | **0** | 0 | 1 | 0 | 0 |
| 1 | **0** | 0 | 0 | 1 | 0 |
| 1 | **0** | 0 | 0 | 0 | 1 |
| $\mathbf{t}^2$ | **1** | 1 | 1 | 1 | 1 |
| $I^2$ | 1 | 1 | 1 | 1 | |

**Iteration 3**

| $\mathbf{a}_3$ | $\mathbf{w}_1^3$ | $\mathbf{v}_1^3$ | $\mathbf{v}_2^3$ | $\mathbf{v}_3^3$ | $\mathbf{w}_2^3$ |
|---|---|---|---|---|---|
| 1 | -1 | **-1** | -1 | -1 | -1 |
| 0 | 0 | **1** | 0 | 0 | 0 |
| 0 | 0 | **0** | 1 | 0 | 0 |
| 1 | 0 | **0** | 0 | 1 | 0 |
| 0 | 0 | **0** | 0 | 0 | 1 |
| $\mathbf{t}^3$ | -1 | **-1** | -1 | 0 | -1 |
| $I^3$ | 1 | 1 | 1 | 1 | 2 |
| | | 2 | 2 | 2 | |

**Iteration 4**

| $\mathbf{a}_4$ | $\mathbf{w}_1^4$ | $\mathbf{w}_2^4$ | $\mathbf{v}_1^4$ | $\mathbf{v}_2^4$ | $\mathbf{w}_3^4$ |
|---|---|---|---|---|---|
| 0 | 0 | -1 | **0** | -1 | 0 |
| 1 | -1 | 1 | **-1** | 0 | -1 |
| 0 | 0 | 0 | **1** | 0 | 0 |
| -1 | 0 | 0 | **0** | 1 | 0 |
| 1 | 0 | 0 | **0** | 0 | 1 |
| $\mathbf{t}^4$ | -1 | 1 | **-1** | -1 | 0 |
| $I^4$ | 1 | 1 | 1 | 1 | 2 |
| | 3 | 2 | 2 | 2 | 3 |
| | | | 3 | 3 | |

**Iteration 5**

| $\mathbf{a}_5$ | $\mathbf{w}_1^5$ | $\mathbf{w}_2^5$ | $\mathbf{w}_3^5$ | $\mathbf{v}_1^5$ | $\mathbf{w}_4^5$ |
|---|---|---|---|---|---|
| 1 | 0 | -1 | 0 | **-1** | 0 |
| 0 | 0 | 0 | -1 | **1** | -1 |
| -1 | -1 | 1 | 1 | **-1** | 0 |
| 1 | 0 | 0 | 0 | **1** | 0 |
| 0 | 0 | 0 | 0 | **0** | 1 |
| $\mathbf{t}^5$ | 1 | -2 | -1 | **1** | 0 |
| $I^5$ | 1 | 1 | 1 | 1 | 2 |
| | 3 | 2 | 2 | 2 | 3 |
| | 4 | 4 | 3 | 3 | 4 |
| | | | | 4 | |

**Iteration 6**

| $\mathbf{a}_6$ | $\mathbf{w}_1^6$ | $\mathbf{w}_2^6$ | $\mathbf{w}_3^6$ | $\mathbf{w}_4^6$ | $\mathbf{w}_5^6$ |
|---|---|---|---|---|---|
| 0 | 1 | -3 | -1 | 1 | 0 |
| 1 | -1 | 2 | 0 | -1 | -1 |
| 0 | 0 | -1 | 0 | 1 | 0 |
| 0 | -1 | 2 | 1 | -1 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 |
| $\mathbf{t}^6$ | -1 | 2 | 0 | -1 | 0 |
| $I^6$ | 1 | 1 | 1 | 1 | 2 |
| | 3 | 2 | 2 | 2 | 3 |
| | 4 | 4 | 3 | 3 | 4 |
| | 5 | 5 | 5 | 4 | 5 |

**Iteration 7**

| $\mathbf{a}_7$ | $\mathbf{w}_1^7$ | $\mathbf{w}_2^7$ | $\mathbf{w}_3^7$ | $\mathbf{w}_4^7$ | $\mathbf{w}_5^7$ | $\mathbf{w}_6^7$ |
|---|---|---|---|---|---|---|
| 1 | 1 | -1 | 1 | 0 | -1 | -1 |
| 1 | -1 | 0 | -1 | -1 | 0 | 0 |
| -1 | 0 | 0 | 1 | 0 | -1 | 1 |
| 0 | -1 | 1 | -1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 | 0 | 0 |
| $\mathbf{t}^7$ | 0 | -1 | -1 | 1 | 0 | -2 |
| $I^7$ | 1 | 1 | 1 | 2 | 1 | 1 |
| | 3 | 2 | 2 | 3 | 4 | 2 |
| | 4 | 3 | 3 | 4 | 5 | 4 |
| | 5 | 5 | 4 | 5 | 6 | 6 |
| | | 6 | | 6 | | |

**Iteration 8**

| $\mathbf{a}_8$ | $\mathbf{w}_1^8$ | $\mathbf{w}_2^8$ | $\mathbf{w}_3^8$ | $\mathbf{w}_4^8$ | $\mathbf{w}_5^8$ | $\mathbf{w}_6^8$ | $\mathbf{w}_7^8$ | $\mathbf{w}_8^8$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | -1 | 1 | -1 | -1 | -1 | 1 | -1 |
| 2 | -1 | 0 | -1 | 0 | 0 | -1 | -2 | -2 |
| 1 | 0 | 0 | 1 | -1 | 1 | 0 | 1 | 1 |
| 2 | -1 | 1 | -1 | 0 | 0 | 1 | -1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 2 |
| $\mathbf{t}^8$ | -3 | 1 | -2 | -2 | 0 | -1 | -4 | -4 |
| $I^8$ | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| | 3 | 2 | 2 | 4 | 2 | 3 | 3 | 4 |
| | 4 | 3 | 3 | 5 | 4 | 5 | 4 | 6 |
| | 5 | 5 | 4 | 6 | 6 | 6 | 7 | 7 |
| | 7 | 6 | | 7 | | 7 | | |

**Dual Cone**

| | $\mathbf{w}_1$ | $\mathbf{w}_2$ | $\mathbf{w}_3$ | $\mathbf{w}_4$ | $\mathbf{w}_5$ | $\mathbf{w}_6$ | $\mathbf{w}_7$ | $\mathbf{w}_8$ | $\mathbf{w}_9$ | $\mathbf{w}_{10}$ | $\mathbf{w}_{11}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 1 | -1 | -1 | -1 | 1 | -1 | -2 | -1 | -3 | -2 |
| | -1 | -1 | 0 | 0 | -1 | -2 | -2 | -1 | -1 | 0 | -1 |
| | 0 | 1 | -1 | 1 | 0 | 1 | 1 | 0 | 1 | -1 | 0 |
| | -1 | -1 | 0 | 0 | 1 | -1 | 0 | 2 | 1 | 2 | 2 |
| | 0 | 0 | 0 | 0 | 1 | 1 | 2 | 0 | 0 | 0 | 1 |
| $I$ | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | 2 |
| | 3 | 2 | 4 | 2 | 3 | 3 | 4 | 3 | 2 | 5 | 3 |
| | 4 | 3 | 5 | 4 | 5 | 4 | 6 | 5 | 3 | 6 | 5 |
| | 5 | 4 | 6 | 6 | 6 | 7 | 7 | 8 | 8 | 8 | 6 |
| | 7 | | 7 | 8 | 7 | | | | | | 8 |

The dual cone asked for is the one generated by the 11 columns in the final table.

*b*) To obtain the edges of the cone $\mathbf{A}_\pi$ we need to obtain the dual cone of the dual of $\mathbf{A}_\pi$. However, we do not need to repeat the tables, because from the final table we can obtain:

**Cono Dual**

| $\mathbf{a}_1$ | $\mathbf{a}_2$ | $\mathbf{a}_3$ | $\mathbf{a}_4$ | $\mathbf{a}_5$ | $\mathbf{a}_6$ | $\mathbf{a}_7$ | $\mathbf{a}_8$ |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 1 | 1 | 1 | 3 | 1 | 4 |
| 2 | 4 | 2 | 2 | 3 | 4 | 3 | 8 |
| 3 | 5 | 5 | 3 | 5 | 5 | 5 | 9 |
| 4 | 6 | 6 | 4 | 8 | 7 | 6 | 10 |
| 8 | 7 | 8 | 6 | 10 | 10 | 7 | 11 |
| 9 | 9 | 9 | 7 | 11 | 11 | | |
| 10 | 11 | 11 | | | | | |

noindent and since we cannot simplify, the edges of the cone $\mathbf{A}_\pi$ are $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6, \mathbf{a}_7, \mathbf{a}_8$

*c*) The main facets of the cone $\mathbf{A}_\pi$ are those in the final table:

$$\{(1,3,4,5,7),(1,2,3,4),(1,4,5,6,7),(1,2,4,6,8),(2,3,5,6,7),(2,3,4,7)$$

$$(2,4,6,7),(1,3,5,8),(1,2,3,8),(1,5,6,8),(2,3,5,6,8)\}.$$

*d*) The edges of the cone $\mathbf{A}_\pi^p$ are $\mathbf{w}_1$ to $\mathbf{w}_{11}$ in the final table.

*e*) To obtain the dual of the given cone, we do not need the last iteration, because the last generator is not in this cone. Thus, in the columns $\mathbf{w}$ of iteration 8 we have the generators of the cone.

*f*) To know if the given vectors belong to the cone $\mathbf{A}$ it is sufficient to multiply the matrix of given vectors by the matrix in iteration 6, to get:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 & -1 & 1 & 0 \\ -1 & 2 & 0 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 & -1 & 1 & 0 \\ -1 & 2 & 0 & -1 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the vectors to belong to the cone, we must have in columns 1, 2, 3, 4 and 5 less than or equal to zero values and in the column 3 zero values. Consequently, no vector belongs to the cone.

*g*) To know if the given system is compatible, we need the dual of the cone generated by the columns of the coefficient matrix, that can be obtained in the table, at iteration 5, and multiply scalarly by the independent terms vector by the generators, obtaining:

$$\begin{pmatrix} a & 0 & 0 & a & -a \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0, & -a, & 0, & 0, & -a \end{pmatrix},$$

which implies $a \geq 0$, because the fourth generating vector is a $v$ vector.

*h*) To solve this, we use the table in iteration 6. Thus, the solution of the system is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \pi_1 \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix} + \pi_2 \begin{pmatrix} -3 \\ 2 \\ -1 \\ 2 \end{pmatrix} + \pi_3 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \pi_4 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

*i*) The solution of the system of the first three equation can be obtained from the table in iteration 5:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \pi_1 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} + \pi_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \pi_3 \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \rho_1 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$