FIRST EVALUATION OF ALGEBRA I
(September 29th, 2015)

1. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & 2 \\ -1 & 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & -2 & 1 \end{pmatrix} \tag{1}$$

Answer the following questions:

a) Determine the rank of this matrix using the orthogonalization algorithm (6 points).

b) Calculate the inverse, if it exists (8 points) and the determinant (2 points) of the submatrix $\mathbf{B}$ of the rows of matrix (1) which do not depend on the previous rows.

c) Investigate if the vectors $(2,2,1,0,0), (1,0,0,0,0), (1,1,1,1,1), (2,3,1,0,1)$ depend linearly of the rows $1,2,3$ and 4 of matrix $\mathbf{A}$ and if they do, obtain their components with respect to them (5 points).

d) What conditions must be satisfied by the constants a, b, c, d and e for the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ a & b & c & d & e \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & 2 \end{pmatrix}$$

to be invertible (5 points).

e) Obtain the orthogonal subspace to the subspace generated by the rows 1, 2 and 4 of $\mathbf{A}$ (4 points).

f) Give the conditions for the system of equations (10 points)

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a \\ 0 \\ b \end{pmatrix} \tag{2}$$

to be compatible.

g) Solve the system (10 points)

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$$

STUDENT'S FIRST AND LAST NAME .....

1. **Problem 1 solution (50 points):**

(a) (6 points)	(b1) (8 points)	(b2) (2 points)
(c) (5 points)	(d) (5 points)	(e) (4 points)
f (10 points)		g (10 points)

SOLUTIONS TO THE FIRST EVALUATION OF ALGEBRA I:
(September, 29th, 2015)

1. SOLUTION TO PROBLEM 1

a) **Solution:** Using the orthogonalization algorithm, we obtain the following tables:

Iteration 1					
$\mathbf{a}_1$	$\mathbf{v}_1^1$	$\mathbf{v}_2^1$	$\mathbf{v}_3^1$	$\mathbf{v}_4^1$	$\mathbf{v}_5^1$
1	**1**	0	0	0	0
1	**0**	1	0	0	0
0	**0**	0	1	0	0
0	**0**	0	0	1	0
0	**0**	0	0	0	1
$\mathbf{t}^1$	**1**	1	0	0	0

Iteration 2					
$\mathbf{a}_2$	$\mathbf{w}_1^2$	$\mathbf{v}_1^2$	$\mathbf{v}_2^2$	$\mathbf{v}_3^2$	$\mathbf{v}_4^2$
0	1	**-1**	0	0	0
1	0	**1**	0	0	0
0	0	**0**	1	0	0
0	0	**0**	0	1	0
0	0	**0**	0	0	1
$\mathbf{t}^2$	0	**1**	0	0	0

Iteration 3					
$\mathbf{a}_3$	$\mathbf{w}_1^3$	$\mathbf{w}_2^3$	$\mathbf{v}_1^3$	$\mathbf{v}_2^3$	$\mathbf{v}_3^3$
1	1	-1	**0**	0	0
1	0	1	**0**	0	0
1	0	0	**1**	0	0
0	0	0	**0**	1	0
0	0	0	**0**	0	1
$\mathbf{t}^3$	1	0	**1**	0	0

Iteration 4					
$\mathbf{a}_4$	$\mathbf{w}_1^4$	$\mathbf{w}_2^4$	$\mathbf{w}_3^4$	$\mathbf{v}_1^4$	$\mathbf{v}_2^4$
0	1	-1	0	0	**0**
0	0	1	0	0	**0**
0	-1	0	1	0	**0**
0	0	0	0	1	**0**
1	0	0	0	0	**1**
$\mathbf{t}^4$	0	0	0	0	**1**

Iteration 5					
$\mathbf{a}_5$	$\mathbf{w}_1^5$	$\mathbf{w}_2^5$	$\mathbf{w}_3^5$	$\mathbf{v}_1^5$	$\mathbf{w}_4^5$
0	1	-1	0	**0**	0
1	0	1	0	**0**	0
-1	-1	0	1	**0**	0
-1	0	0	0	**1**	0
2	0	0	0	**0**	1
$\mathbf{t}^5$	1	1	-1	**-1**	2

Iteration 6					
$\mathbf{a}_6$	$\mathbf{w}_1^6$	$\mathbf{w}_2^6$	$\mathbf{w}_3^6$	$\mathbf{w}_5^6$	$\mathbf{w}_4^6$
-1	1	-1	0	0	0
0	0	1	0	0	0
0	-1	0	1	0	0
-1	1	1	-1	-1	2
2	0	0	0	0	1
$\mathbf{t}^6$	-2	0	1	1	0

Iteration 7					
$\mathbf{a}_7$	$\mathbf{w}_1^7$	$\mathbf{w}_2^7$	$\mathbf{w}_3^7$	$\mathbf{w}_5^7$	$\mathbf{w}_4^7$
0	1	-1	0	0	0
0	0	1	0	0	0
-1	-1	0	1	0	0
-2	1	1	-1	-1	2
1	0	0	0	0	1
$\mathbf{t}^7$	-1	-2	1	2	-3

FINAL TABLE					
	$\mathbf{w}_1$	$\mathbf{w}_2$	$\mathbf{w}_3$	$\mathbf{w}_5$	$\mathbf{w}_4$
	1	-1	0	0	0
	0	1	0	0	0
	-1	0	1	0	0
	1	1	-1	-1	2
	0	0	0	0	1

b) Since we could pivot five times, the rank is 5.

c) The inverse matrix exists because the rank is 5. The inverse matrix can be obtained from the final table changing the order of the columns according to the pivot columns, that is:

$$B^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The determinant is 1, that is, the product of all pivots, 1, times $(-1)^1$, because we need to perform one permutation, due to the selected pivoting.

d) If we multiply the vector matrix by the matrix in iteration 6, we have:

$$\begin{pmatrix} 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Since the fourth column has a non null value in row 3, only the third vector is not linearly dependent of the rows $1, 2, 3$ and 4 of $\mathbf{A}$.

The coefficients to be used by the linear combinations are in columns 1,2,3 and 5, which correspond to the pivot columns.

e) It is sufficient to multiply the vector (a, b, c, d, e) by the vector in column 2 of the inverse matrix in question and force it to be non null. Then, we get:

$$(a, b, c, d, e) \bullet (-1, 1, 0, 1, 0) = -a + b + d \neq 0.$$

f) To obtain the demanded orthogonal subspace we must remove the columns used as pivots (1, 2 and 5) in the table. Then, we get:

$$\mathcal{L}((0, 0, 1, -1, 0), (0, 0, 0, -1, 0)).$$

g) To study the compatibility, we need to obtain the orthogonal subspace to the subspace generated by the column vectors of the coefficients. This has been done in the table, to get (see table in iteration 6 or 7, after eliminating the columns 1, 3 and 4):

$$\mathcal{L}((-1,1,0,1,0),(0,0,0,2,1))$$

If we multiply scalarly these vectors by the column vector of independent terms and force it to be zero, we get

$$b = 0.$$

h) To solve this system we need to eliminate columns 1, 3 and 4 in iterations 6 or 7 and make $x_5 = 1$. This leads to:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} + \rho \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$