Complete linear systems of equations

Algebra

SOLVING A COMPLETE LINEAR SYSTEM

$$
\begin{array}{lllllll}
a_{11}x_1 & +a_{12}x_2 & + & \cdots & +a_{1n}x_n & = & b_1 \\
a_{21}x_1 & +a_{22}x_2 & + & \cdots & +a_{2n}x_n & = & b_2 \\
\cdots & \cdots & + & \cdots & \cdots & \cdots & \cdots \\
a_{m1}x_1 & +a_{m2}x_2 & + & \cdots & +a_{mn}x_n & = & b_m
\end{array}
$$

$$
\begin{array}{lllllll}
a_{11}x_1 & +a_{12}x_2 & + & \cdots & +a_{1n}x_n & -b_1x_{n+1} & = & 0 \\
a_{21}x_1 & +a_{22}x_2 & + & \cdots & +a_{2n}x_n & -b_2x_{n+1} & = & 0 \\
\cdots & \cdots & + & \cdots & \cdots & \cdots & \cdots & \\
a_{m1}x_1 & +a_{m2}x_2 & + & \cdots & +a_{mn}x_n & -b_mx_{n+1} & = & 0 \\
 & & & & & x_{n+1} & = & 1
\end{array}
$$

A complete linear system is solved by solving the homogeneous system and the condition

$$x_{n+1} = 1.$$

$$
\begin{array}{rrrrcr}
 & x_2 & -x_3 & +2x_4 & = & -1 \\
x_1 & +x_2 & +x_3 & & = & 0 \\
x_1 & +2x_2 & & +x_4 & = & -1 \\
x_1 & +x_2 & & +x_4 & = & 0
\end{array}
\Leftrightarrow
\begin{array}{rrrrrcr}
 & x_2 & -x_3 & +2x_4 & +x_5 & = & 0 \\
x_1 & +x_2 & +x_3 & & & = & 0 \\
x_1 & +2x_2 & & +x_4 & +x_5 & = & 0 \\
x_1 & +x_2 & & +x_4 & & = & 0 \\
 & & & & x_5 & = & 1
\end{array}
$$

	Iteration 1				
a_1	v_1	$\mathbf{v_2}$	v_3	v_4	v_5
0	1	**0**	0	0	0
1	0	**1**	0	0	0
-1	0	**0**	1	0	0
2	0	**0**	0	1	0
1	0	**0**	0	0	1
t	0	1	-1	2	1

	Iteration 2				
a_2	$\mathbf{v_1}$	w_1	v_3	v_4	v_5
1	**1**	0	0	0	0
1	**0**	1	1	-2	-1
1	**0**	0	1	0	0
0	**0**	0	0	1	0
0	**0**	0	0	0	1
t	1	1	2	-2	-1

	Iteration 3				
a_3	w_2	w_1	v_3	$\mathbf{v_4}$	v_5
1	1	-1	-2	**2**	1
2	0	1	1	**-2**	-1
0	0	0	1	**0**	0
1	0	0	0	**1**	0
1	0	0	0	**0**	1
t	1	1	0	-1	0

	Iteration 4				
a_4	w_2	w_1	$\mathbf{v_3}$	w_3	v_5
1	3	1	**-2**	-2	1
1	-2	-1	**1**	2	-1
0	0	0	**1**	-0	0
1	1	1	**0**	-1	0
0	0	0	**0**	-0	1
t	2	1	-1	-1	0

	Final table				
	w_2	w_1	w_4	w_3	v_5
	-1	-1	2	0	1
	0	0	-1	1	-1
	2	1	-1	-1	0
	1	1	-0	-1	0
	0	0	-0	-0	1

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \rho \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}
$$

If you do not want to solve subsystems, pivot columns can be removed as soon as you use them, saving calculations.

How to impose the condition x5 = 1?

To do this proceed as follows:
The solution of the homogeneous system that has been solved is the subspace orthogonal to the generated by the 4 vectors corresponding to the coefficients of the 4 equations, so we must eliminate the columns with which it has been pivoted, leaving only the last one, so this solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \rho \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \rho \\ -\rho \\ 0 \\ 0 \\ \rho \end{pmatrix}.$$

To satisfy the condition $x_5 = 1$ it must be $\rho_3 = 1$, so by also eliminating the component x_5, that was created fictitiously, it turns out

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

what proves that the initial system has a unique solution.

Let's see now what it would be like if we wanted to solve the system of the first two equations. To do this, the subspace orthogonal to the subspace associated with these equations should be obtained, for which the columns used to pivot should be eliminated in the final table, which were the second and the first, in this order, with which the solution of the homogeneous system , including x5 would be:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \rho_1 \begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \rho_2 \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} + \rho_3 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\rho_1 + \rho_3 \\ -\rho_1 + \rho_2 - \rho_3 \\ -\rho_1 - \rho_2 \\ -\rho_2 \\ \rho_3 \end{pmatrix}.$$

To satisfy the condition $x_5 = 1$ it must be $\rho_3 = 1$, so by also eliminating the component x_5, that was created fictitiously, it turns out

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \rho_1 \begin{pmatrix} 2 \\ -1 \\ -1 \\ 0 \end{pmatrix} + \rho_2 \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\rho_1 + 1 \\ -\rho_1 + \rho_2 - 1 \\ -\rho_1 - \rho_2 \\ -\rho_2 \end{pmatrix}$$

what proves that the system has a set of infinite solutions that consists of a vector space of dimension 2 and a particular solution, which is the same as the previous one.

Finally, it is interesting to see that, despite the dependence of the system solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2\rho_1 + 1 \\ -\rho_1 + \rho_2 - 1 \\ -\rho_1 - \rho_2 \\ -\rho_2 \end{pmatrix}$$

of two arbitrary values real numbers ρ_1 and ρ_2), it is satisfied identically, that is, by replacing these solutions into the two equations:

$$\begin{array}{rrrrcr} & x_2 & -x_3 & +2x_4 & = & -1 \\ x_1 & +x_2 & +x_3 & & = & 0 \end{array}$$

all terms including ρ_1 and ρ_2 cancel and the equations hold. Let s see:

$$x_2 - x_3 + 2x_4 = (-\rho_1 + \rho_2 - 1) - (-\rho_1 - \rho_2) + 2(-\rho_2) = -1$$
$$x_1 + x_2 + x_3 = 2\rho_1 + 1 + (-\rho_1 + \rho_2 - 1) + (-\rho_1 - \rho_2) = 0.$$

Therefore, the method not only solves a system of equations, but all its possible subsystems.

$$\begin{array}{rrrrcr} & x_2 & -x_3 & +2x_4 & = & -1 \\ x_1 & +x_2 & +x_3 & & = & 0 \\ x_1 & +2x_2 & & +x_4 & = & -1 \\ x_1 & +x_2 & & +x_4 & = & 0 \end{array}$$

Final table

w_2	w_1	w_4	w_3	v_5
-1	-1	2	0	1
0	0	-1	1	-1
2	1	-1	-1	0
1	1	-0	-1	0
0	0	-0	-0	1

System	Solution
$\begin{array}{rrrrcr} & x_2 & -x_3 & +2x_4 & = & -1 \\ x_1 & +x_2 & +x_3 & & = & 0 \\ x_1 & +2x_2 & & +x_4 & = & -1 \\ x_1 & +x_2 & & +x_4 & = & 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$
$x_1 \ +x_2 \quad +x_4 \ = \ 0$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \rho_1 \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + \rho_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \rho_3 \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$
$\begin{array}{rrrrcr} x_1 & +2x_2 & & +x_4 & = & -1 \\ x_1 & +x_2 & & +x_4 & = & 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \rho_1 \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + \rho_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$
$\begin{array}{rrrrcr} & x_2 & -x_3 & +2x_4 & = & -1 \\ x_1 & +2x_2 & & +x_4 & = & -1 \\ x_1 & +x_2 & & +x_4 & = & 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \rho_1 \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

$$
\begin{array}{rrrrcr}
 & x_2 & -x_3 & +2x_4 & = & -1 \\
x_1 & +x_2 & +x_3 & & = & 0 \\
x_1 & +2x_2 & & +2x_4 & = & 0 \\
x_1 & +x_2 & & +x_4 & = & 0
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{rrrrrcr}
 & x_2 & -x_3 & +2x_4 & +x_5 & = & 0 \\
x_1 & +x_2 & +x_3 & & & = & 0 \\
x_1 & +2x_2 & & +2x_4 & & = & 0 \\
x_1 & +x_2 & & +x_4 & & = & 0 \\
 & & & & x_5 & = & 1
\end{array}
$$

	Iteration 1				
a_1	v_1	$\mathbf{v_2}$	v_3	v_4	v_5
0	1	**0**	0	0	0
1	0	**1**	0	0	0
-1	0	**0**	1	0	0
2	0	**0**	0	1	0
1	0	**0**	0	0	1
t	0	1	-1	2	1

	Iteration 2				
a_2	$\mathbf{v_1}$	w_1	v_3	v_4	v_5
1	**1**	0	0	0	0
1	**0**	1	1	-2	-1
1	**0**	0	1	0	0
0	**0**	0	0	1	0
0	**0**	0	0	0	1
t	1	1	2	-2	-1

	Iteration 3				
a_3	w_2	w_1	v_3	v_4	$\mathbf{v_5}$
1	1	-1	-2	2	**1**
2	0	1	1	-2	**-1**
0	0	0	1	0	**0**
2	0	0	0	1	**0**
0	0	0	0	0	**1**
t	1	1	0	0	-1

	Iteration 4				
a_4	w_2	w_1	$\mathbf{v_3}$	v_4	w_3
1	2	0	**-2**	2	-1
1	-1	0	**1**	-2	1
0	0	0	**1**	0	-0
1	0	0	**0**	1	-0
0	1	1	**0**	0	-1
t	1	0	-1	1	0

	Final table				
	w_2	w_1	w_4	v_4	w_3
	0	0	2	0	-1
	0	0	-1	-1	1
	1	0	-1	1	0
	0	0	-0	1	0
	1	1	-0	0	-1

THERE IS NO SOLUTION BECAUSE YOU HAVE TO PIVOT WITH THE LAST COLUMN AND THE LAST COMPONENTS OF OTHER COLUMNS ARE NULL