Inverse of a matrix

Algebra

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 2 & 1 & 0 \end{pmatrix}$$

Iteration 1

a_1	v_1	v_2	v_3	v_4
1	1	0	0	0
1	0	1	0	0
0	0	0	1	0
1	0	0	0	1
t	1	1	0	1

Iteration 2

a_2	w_1	v_2	v_3	v_4
0	1	-1	0	-1
1	0	1	0	0
0	0	0	1	0
1	0	0	0	1
t	0	1	0	1

Iteration 3

a_3	w_1	w_2	v_3	v_4
0	1	-1	0	0
0	0	1	0	-1
-1	0	0	1	0
1	0	0	0	1
t	0	0	-1	1

Iteration 4

a_4	w^1	w^2	w^3	v^4
-1	1	-1	0	0
2	0	1	0	-1
1	0	0	-1	1
0	0	0	0	1
t	-1	3	-1	-1

Iteration 5

	w_1	w_2	w_3	w_4
	1	-1	0	0
	1	-2	1	1
	-1	3	-2	-1
	-1	3	-1	-1

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 2 & 1 & 0 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ -1 & 3 & -2 & -1 \\ -1 & 3 & -1 & -1 \end{pmatrix}$$

Iteration 1				
a_1	v_1	v_2	v_3	v_4
1	1	0	0	0
1	0	1	0	0
0	0	0	1	0
1	0	0	0	1
t	1	1	0	1

Iteration 2				
a_2	w_1	v_2	v_3	v_4
0	1	-1	0	-1
1	0	1	0	0
0	0	0	1	0
1	0	0	0	1
t	0	1	0	1

Iteration 3				
a_3	w_1	w_2	v_3	v_4
0	1	-1	0	0
0	0	1	0	-1
-1	0	0	1	0
1	0	0	0	1
t	0	0	-1	1

Iteration 4				
a_4	w^1	w^2	w^3	v^4
-1	1	-1	0	0
2	0	1	0	-1
1	0	0	-1	1
0	0	0	0	1
t	-1	3	-1	-1

Iteration 5				
	w_1	w_2	w_3	w_4
	1	-1	0	0
	1	-2	1	1
	-1	3	-2	-1
	-1	3	-1	-1

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 2 & 1 & 0 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ -1 & 3 & -2 & -1 \\ -1 & 3 & -1 & -1 \end{pmatrix}$$

Iteration 1

a_1	v_1	v_2	v_3	v_4
1	1	0	0	0
1	0	1	0	0
0	0	0	1	0
1	0	0	0	1
t	1	1	0	1

Iteration 2

a_2	w_1	v_2	v_3	v_4
0	1	-1	0	-1
1	0	1	0	0
0	0	0	1	0
1	0	0	0	1
t	0	1	0	1

Iteration 3

a_3	w_1	w_2	v_3	v_4
0	1	-1	0	0
0	0	1	0	-1
-1	0	0	1	0
1	0	0	0	1
t	0	0	-1	1

Iteration 4

a_4	w^1	w^2	w^3	v^4
-1	1	-1	0	0
2	0	1	0	-1
1	0	0	-1	1
0	0	0	0	1
t	-1	3	-1	-1

Iteration 5

	w_1	w_2	w_3	w_4
	1	-1	0	0
	1	-2	1	1
	-1	3	-2	-1
	-1	3	-1	-1

$$|A| = (-1)_0 \times 1 \times 1 \times (-1) \times (-1) = 1$$

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 2 & 1 & 0 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ -1 & 3 & -2 & -1 \\ -1 & 3 & -1 & -1 \end{pmatrix}$$

Iteration 1				
a_1	v_1	v_2	v_3	v_4
1	1	0	0	0
1	0	1	0	0
0	0	0	1	0
1	0	0	0	1
t	1	1	0	1

Iteration 2				
a_2	w_1	v_2	v_3	v_4
0	1	-1	0	-1
1	0	1	0	0
0	0	0	1	0
1	0	0	0	1
t	0	1	0	1

Iteration 3				
a_3	w_1	w_2	v_3	v_4
0	1	-1	0	0
0	0	1	0	-1
-1	0	0	1	0
1	0	0	0	1
t	0	0	-1	1

Iteration 4				
a_4	w^1	w^2	w^3	v^4
-1	1	-1	0	0
2	0	1	0	-1
1	0	0	-1	1
0	0	0	0	1
t	-1	3	-1	-1

Iteration 5				
	w_1	w_2	w_3	w_4
	1	-1	0	0
	1	-2	1	1
	-1	3	-2	-1
	-1	3	-1	-1

$$|A| = (-1)^0 \times 1 \times 1 \times (-1) \times (-1) = 1$$

Determinant $|A| = |B|^{-1}(-1)^p \prod_{i=1}^{n} t_i$

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} \\ -1 & 2 & 1 & 0 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ -1 & 3 & -2 & -1 \\ -1 & 3 & -1 & -1 \end{pmatrix}$$

$$A^{*} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{3} \\ -1 & 2 & 1 & 0 \end{pmatrix}$$

$$A^{*-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & -1 & 0 \\ -1 & -5 & 2 & 1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$$

Iteration 1

a_1	v_1	v_2	v_3	v_4
1	1	0	0	0
1	0	1	0	0
0	0	0	1	0
1	0	0	0	1
t	1	1	0	1

Iteration 2

a_2	w_1	v_2	v_3	v_4
0	1	-1	0	-1
1	0	1	0	0
0	0	0	1	0
1	0	0	0	1
t	0	1	0	1

Iteration 3

a_3	w_1	w_2	v_3	v_4
0	1	-1	0	0
0	0	1	0	-1
-1	0	0	1	0
1	0	0	0	1
t	0	0	-1	1

Iteration 4

a_4	w^1	w^2	w^3	v^4
-1	1	-1	0	0
2	0	1	0	-1
1	0	0	-1	1
0	0	0	0	1
t	-1	3	-1	-1

Iteration 5

a_5	w_1	w_2	w_3	w_4
1	1	-1	0	0
2	1	-2	1	1
0	-1	3	-2	-1
3	-1	3	-1	-1
t	0	4	$\boxed{-1}$	-1

New inverse

	w^1	w^2	w^5	w^4
	1	-1	0	0
	1	2	-1	0
	-1	-5	2	1
	-1	-1	1	0

UPDATE DETERMINANT : |A*| = |A| x (-1) = (1) x (-1) = -1

You have to pivot with the column that occupies the same place as the changed row

Initial matrix and its inverse

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} \\ -1 & 2 & 1 & 0 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ -1 & 3 & -2 & -1 \\ -1 & 3 & -1 & -1 \end{pmatrix}$$

Inverse				
a_5	w_1	w_2	w_3	w_4
1	1	-1	0	0
2	1	-2	1	1
0	-1	3	-2	-1
3	-1	3	-1	-1
t	0	4	$\boxed{-1}$	-1

New Inverse				
	w^1	w^2	w^3	w^4
	1	-1	0	0
	1	2	-1	0
	-1	-5	2	1
	-1	-1	1	0

Updated matrix and its inverse

$$A^* = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{3} \\ -1 & 2 & 1 & 0 \end{pmatrix} \qquad A^{*-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & -1 & 0 \\ -1 & -5 & 2 & 1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$$

You have to pivot with the column that occupies the same place as the changed row

Obtaining simultaneous diagonal inverses

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 2 & 1 & 0 \end{pmatrix}$$

Iteration 1				
a_1	v_1	v_2	v_3	v_4
1	1	0	0	0
1	0	1	0	0
0	0	0	1	0
1	0	0	0	1
t	1	1	0	1

Iteration 2				
a_2	w_1	v_2	v_3	v_4
0	1	-1	0	-1
1	0	1	0	0
0	0	0	1	0
1	0	0	0	1
t	0	1	0	1

Iteration 3				
a_3	w_1	w_2	v_3	v_4
0	1	-1	0	0
0	0	1	0	-1
-1	0	0	1	0
1	0	0	0	1
t	0	0	-1	1

Iteration 4				
a_4	w^1	w^2	w^3	v^4
-1	1	-1	0	0
2	0	1	0	-1
1	0	0	-1	1
0	0	0	0	1
t	-1	3	-1	-1

Iteration 5				
	w_1	w_2	w_3	w_4
	1	-1	0	0
	1	-2	1	1
	-1	3	-2	-1
	-1	3	-1	-1

Determinant : $|A| = |B|^{-1}(-1)^{p}\prod_{i=1}^{n} t_i$

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$

	Iteration 1			
a_1	$\mathbf{v}_1$	v_2	v_3	v_4
1	**1**	0	0	0
1	**0**	1	0	0
1	**0**	0	1	0
0	**0**	0	0	1
t	1	1	1	0

	Iteration 2			
a_2	w_1	v_2	$\mathbf{v}_3$	v_4
1	1	-1	**-1**	0
1	0	1	**0**	0
0	0	0	**1**	0
0	0	0	**0**	1
t	1	0	-1	0

	Iteration 3			
a_3	w_1	v_2	w_2	$\mathbf{v}_4$
0	0	-1	1	**0**
0	0	1	-0	**0**
1	1	0	-1	**0**
1	0	0	-0	**1**
t	1	0	-1	1

	Iteration 4			
a_4	w_1	$\mathbf{v}_2$	w_2	w_3
-1	0	**-1**	1	0
0	0	**1**	0	0
0	1	**0**	-1	0
-2	-1	**0**	1	1
t	2	1	-3	-2

	Final table			
	w_1	w_4	w_2	w_3
	2	-1	-2	-2
	-2	1	3	2
	1	0	-1	0
	-1	0	1	1

	Permuted final table			
	w_1	w_2	w_3	w_4
	2	-2	-2	-1
	-2	3	2	1
	1	-1	0	0
	-1	1	1	0

$$\begin{matrix} 1 & 3 & 4 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 2 & 3 & \end{matrix}$$

$$A^{-1} = \begin{pmatrix} 2 & -2 & -2 & -1 \\ -2 & 3 & 2 & 1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 1 & 0 \end{pmatrix}$$

This implies two permutations

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 2 & 1 & 0 \end{pmatrix}$$

Iteration 1

a_1	v_1	v_2	v_3	v_4
1	1	0	0	0
1	0	1	0	0
0	0	0	1	0
1	0	0	0	1
t	1	1	0	1

Iteration 2

$$A_1 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

a_2	w_1	v_2	v_3	v_4
0	1	-1	0	-1
1	0	1	0	0
0	0	0	1	0
1	0	0	0	1
t	0	1	0	1

Iteration 3

$$A_2 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

a_3	w_1	w_2	v_3	v_4
0	1	-1	0	0
0	0	1	0	-1
-1	0	0	1	0
1	0	0	0	1
t	0	0	-1	1

Iteration 4

$$A_3 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

a_4	w_1	w_2	w_3	v_4
-1	1	-1	0	0
2	0	1	0	-1
1	0	0	-1	1
0	0	0	0	1
t	-1	3	-1	-1

Iteration 5

$$A_4 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 2 & 1 & 0 \end{pmatrix}$$

	w_1	w_2	w_3	w_4
	1	-1	0	0
	1	-2	1	1
	-1	3	-2	-1
	-1	3	-1	-1

Problem

Which elements of an invertible matrix can be changed arbitrarily without ceasing to be invertible?

Would the determinant change after these changes?

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Which elements of an invertible matrix can be changed arbitrarily without ceasing to be invertible?

Would the determinant change after these changes?

Let a_{ij} be the elements of the initial matrix, a_{pq} be the element changed by x and assume that the inverse matrix in its column p has elements $\{b_{1p}, b_{2p}, \ldots, b_{np}\}$, as it is shown below

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1q} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2q} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{p1} & a_{p2} & \cdots & \mathbf{a_{pq}} & \cdots & a_{pn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nq} & \cdots & a_{nn} \end{pmatrix} \quad ; \quad \mathbf{A}^{-1} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2p} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{i1} & b_{i2} & \cdots & b_{qp} & \cdots & b_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{np} & \cdots & b_{nn} \end{pmatrix}$$

$$\mathbf{A}^{*} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1q} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2q} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{p1} & a_{p2} & \cdots & \mathbf{x} & \cdots & a_{pn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nq} & \cdots & a_{nn} \end{pmatrix}$$

Then, the solution to these problems is immediate if we take into account all of the above, by simply considering the following:

1. *The new determinant of the matrix when changing an element is the previous one multiplied by the new scalar product of the changed row.*

2. *The scalar product of this row before changing is:*

$$\sum_{j=1}^{n} a_{pj} b_{jp} = 1 .$$

3. *The new scalar product, after changing the element, is equal to*

$$\sum_{j=1, j\neq q}^{n} a_{pj} b_{jp} + x b_{qp} = 1 - a_{pq} b_{qp} + x b_{qp} = 1 - b_{qp}(a_{pq} - x),$$

   that should be non-zero for every possible value of x, which implies $b_{qp} = 0$.

4. *If* $b_{qp} = 0$ *the determinant of the new matrix coincides with the previous one, since the new scalar product is equal to the unit. That is, in this case, the value of the determinant is not altered.*

Therefore, the only elements aij that can be arbitrarily changed in an invertible matrix to remain invertible are those corresponding to the null values bji = 0 of the inverse matrix.

Example 1 (Invertibility after change of element in an invertible matrix) Let the matrix A given below be, if its inverse is calculated and transposed it results:

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 3 & -5 \\ 0 & -1 & -1 & 3 \\ 0 & -1 & -2 & 3 \\ 1 & -3 & -4 & 7 \end{pmatrix} \; ; \quad \mathbf{A}^{-1} = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 3 & 1 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \; ; \left(\mathbf{A}^T\right)^{-1} = \begin{pmatrix} 2 & 3 & \mathbf{0} & 1 \\ \mathbf{0} & 1 & 1 & 1 \\ 1 & 4 & -1 & 1 \\ 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

If the elements of A that occupy the positions of the null elements in the transposed matrix of the inverse by the value x are changed one by one, the five matrices are obtained:

$$\begin{pmatrix} 0 & 2 & \mathbf{x} & -5 \\ 0 & -1 & -1 & 3 \\ 0 & -1 & -2 & 3 \\ 1 & -3 & -4 & 7 \end{pmatrix} \; ; \quad \begin{pmatrix} 0 & 2 & 3 & -5 \\ \mathbf{x} & -1 & -1 & 3 \\ 0 & -1 & -2 & 3 \\ 1 & -3 & -4 & 7 \end{pmatrix} \; ; \quad \begin{pmatrix} 0 & 2 & 3 & -5 \\ 0 & -1 & -1 & 3 \\ 0 & -1 & -2 & 3 \\ 1 & \mathbf{x} & -4 & 7 \end{pmatrix} ;$$

$$\begin{pmatrix} 0 & 2 & 3 & -5 \\ 0 & -1 & -1 & 3 \\ 0 & -1 & -2 & 3 \\ 1 & -3 & \mathbf{x} & 7 \end{pmatrix} \; ; \quad \textbf{and} \quad \begin{pmatrix} 0 & 2 & 3 & -5 \\ 0 & -1 & -1 & 3 \\ 0 & -1 & -2 & 3 \\ 1 & -3 & -4 & \mathbf{x} \end{pmatrix}$$

whose determinant is the same as that of matrix A, so they are invertible regardless of the value of x.