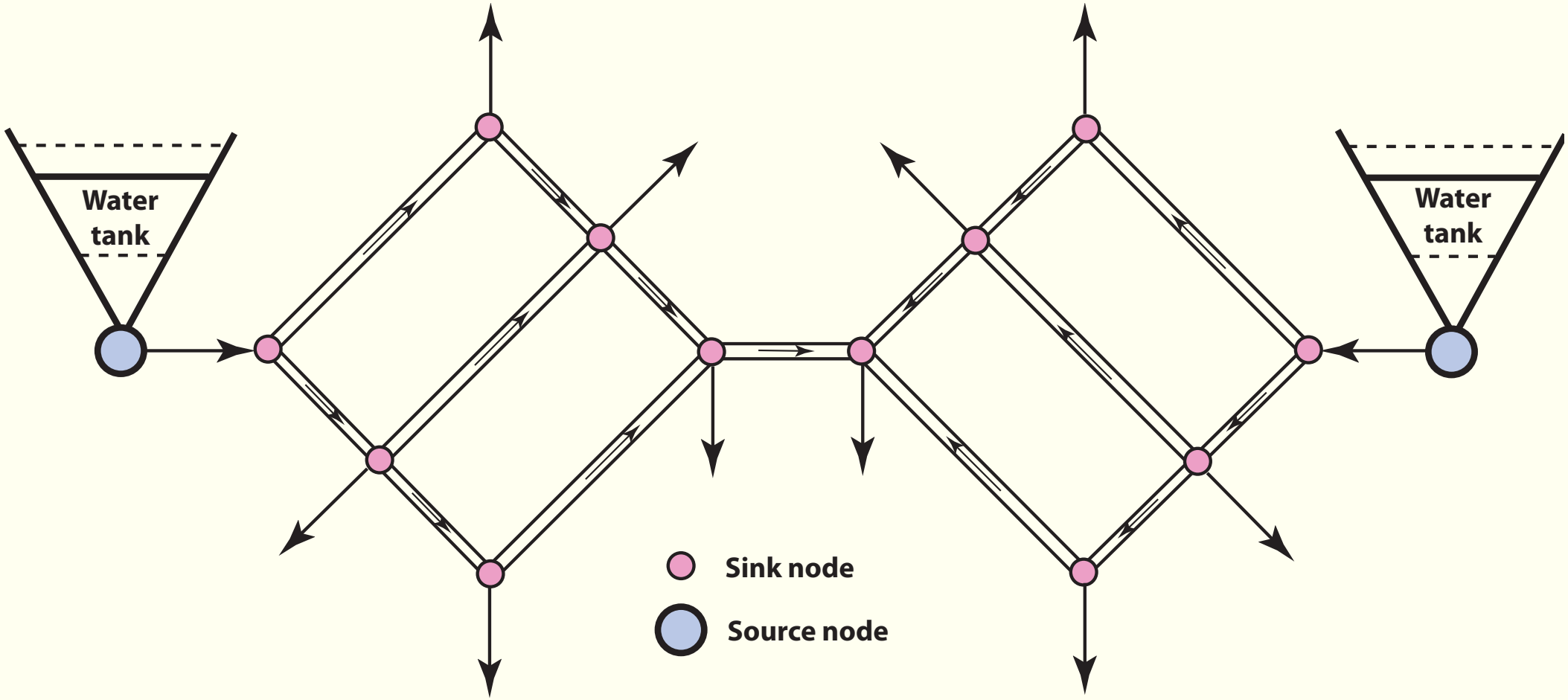
Examples of applications.
The water supply problem.

Algebra

THE WATER SUPPLY PROBLEM



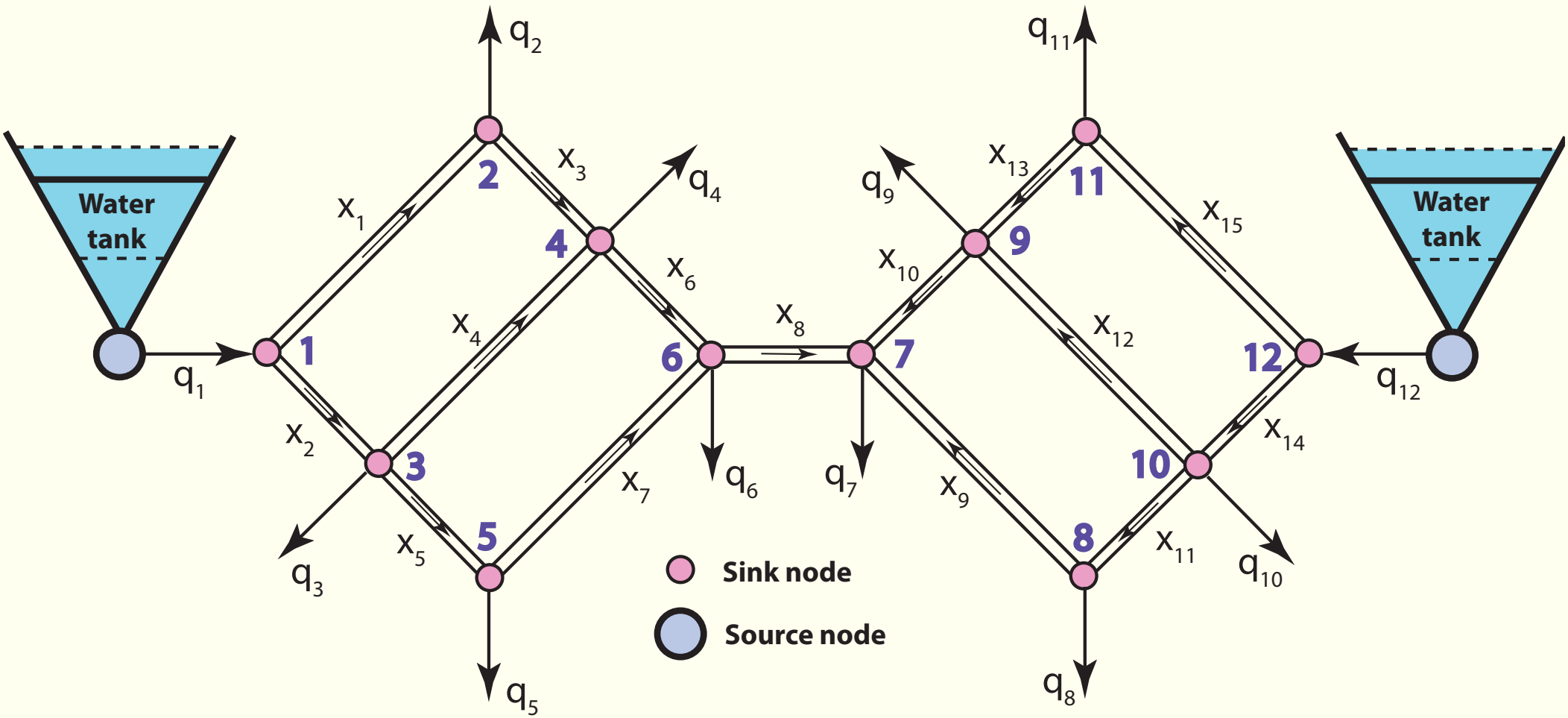
The student must identify the unknowns of the problem

The student must identify the equations of the problem

THE WATER SUPPLY PROBLEM

What are the data?

What are the unknowns?

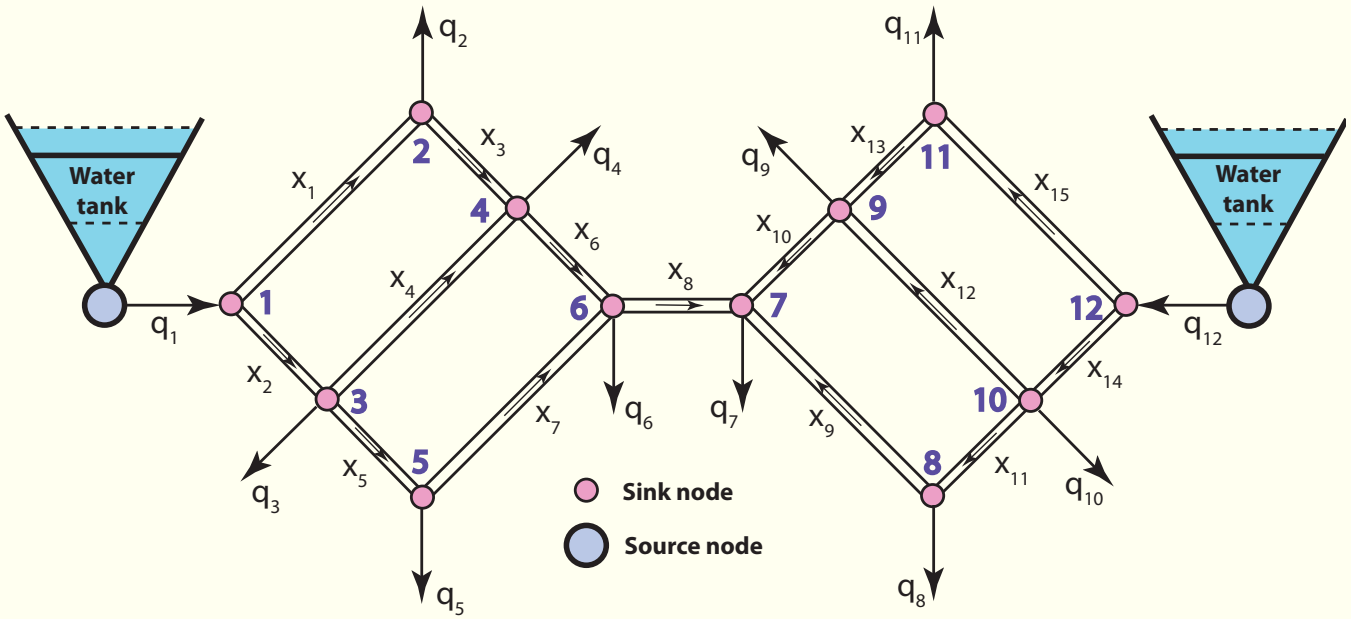


Number of equations.
Number of unknowns.

Node numbering.
Constraints.

STATEMENT OF THE PROBLEM

Problem Statement

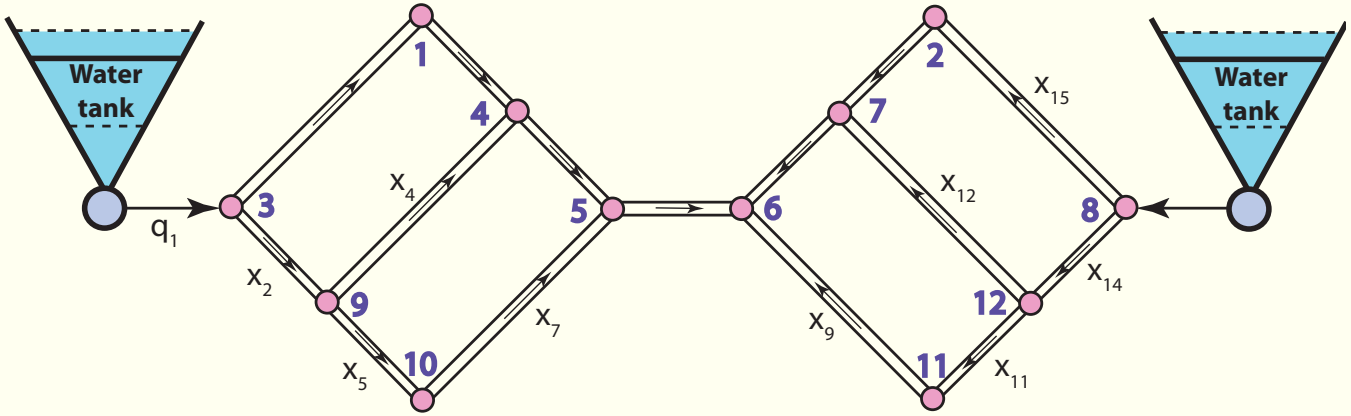


$$
\begin{bmatrix}
-1 & -1 & & & & & & & & & & & & & \\
1 & & -1 & & & & & & & & & & & & \\
& 1 & & -1 & -1 & & & & & & & & & & \\
& & 1 & 1 & & -1 & & & & & & & & & \\
& & & & 1 & & -1 & & & & & & & & \\
& & & & & 1 & 1 & -1 & & & & & & & \\
& & & & & & & 1 & 1 & 1 & & & & & \\
& & & & & & & & -1 & & 1 & & & & \\
& & & & & & & & & -1 & & 1 & 1 & & \\
& & & & & & & & & & -1 & -1 & & 1 & \\
& & & & & & & & & & & & -1 & & 1 \\
& & & & & & & & & & & & & -1 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15}
\end{bmatrix}
=
\begin{bmatrix}
-q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ q_{11} \\ -q_{12}
\end{bmatrix}
$$

The student must know how to state the problem in the form of a system of linear equations and specially in matrix form.

STATEMENT OF THE PROBLEM

Statement of the problem

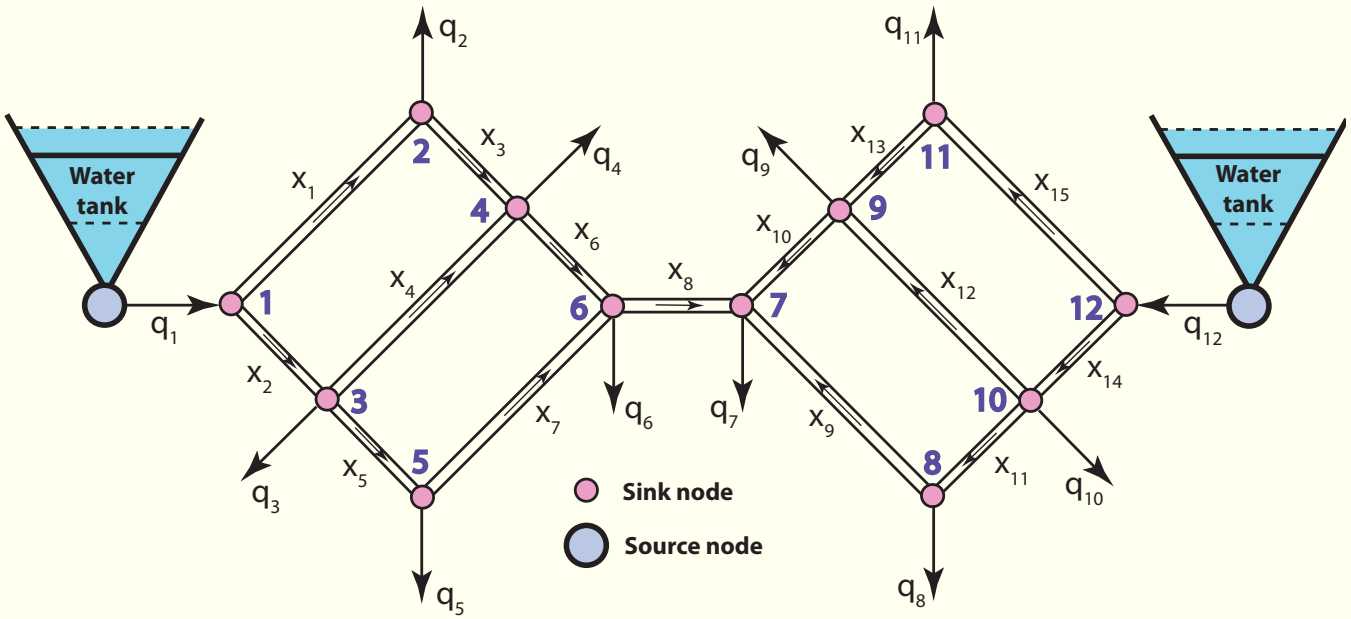


$$
\begin{bmatrix}
1 & & -1 & & & & & & & & & & & & \\
 & & & & & & & & & & & & -1 & & 1 \\
-1 & -1 & & & & & & & & & & & & & \\
 & & 1 & 1 & & -1 & & & & & & & & & \\
 & & & & & 1 & 1 & -1 & & & & & & & \\
 & & & & & & & 1 & 1 & 1 & & & & & \\
 & & & & & & & & & -1 & & 1 & 1 & & \\
 & & & & & & & & & & & & & -1 & -1 \\
 & 1 & & -1 & -1 & & & & & & & & 1 & & \\
 & & & & 1 & & -1 & & & & & & & & \\
 & & & & & & & & -1 & & 1 & & & & \\
 & & & & & & & & & & -1 & -1 & & 1 &
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{bmatrix}
=
\begin{bmatrix} -q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ q_{11} \\ -q_{12} \end{bmatrix}
$$

The student must know how to number the nodes and differentiate between a convenient numbering and one that is not.

STATEMENT OF THE PROBLEM

Analysis of the solution

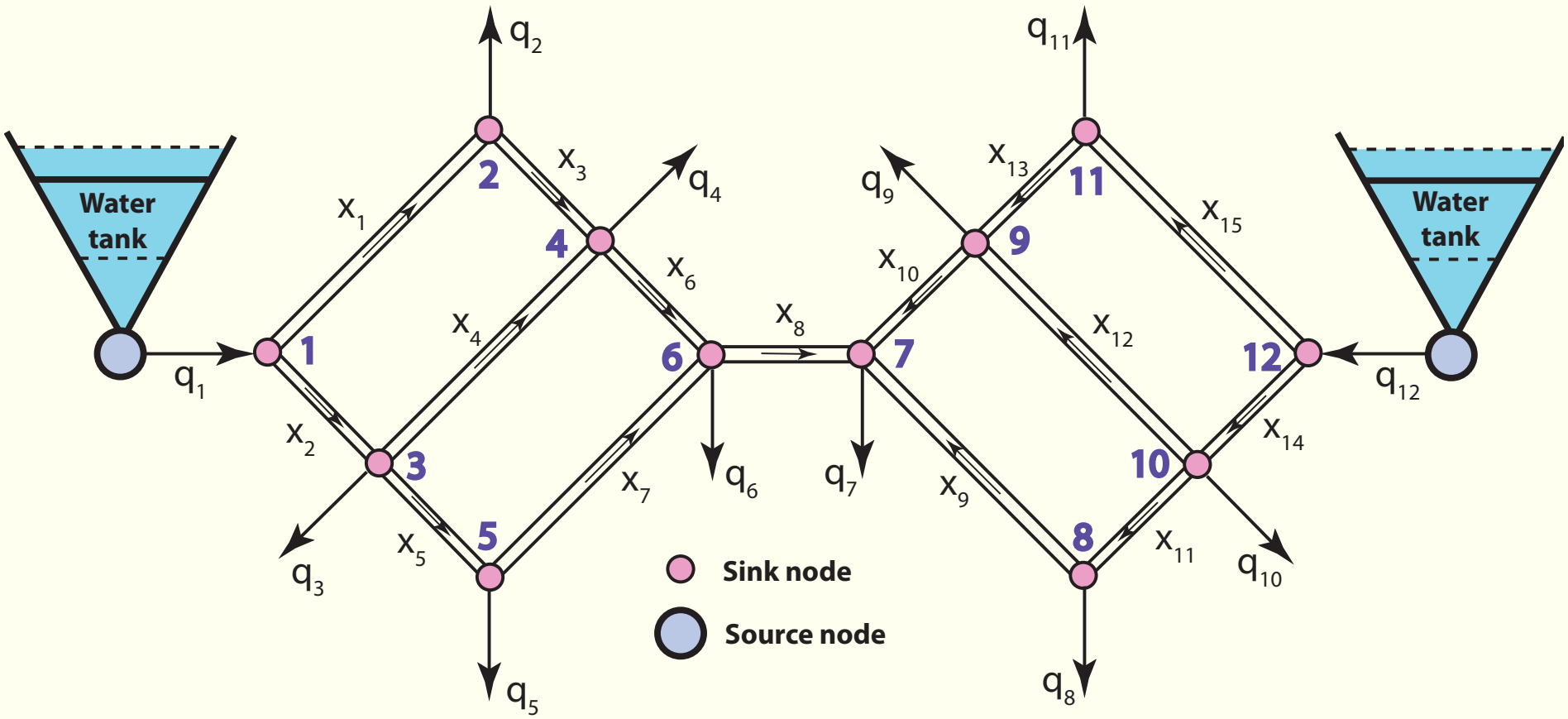


$$
\begin{bmatrix}
-1 & -1 & & & & & & & & & & & & & \\
1 & & -1 & & & & & & & & & & & & \\
 & 1 & & -1 & -1 & & & & & & & & & & \\
 & & 1 & 1 & & -1 & & & & & & & & & \\
 & & & & 1 & & -1 & & & & & & & & \\
 & & & & & 1 & 1 & -1 & & & & & & & \\
 & & & & & & & 1 & 1 & 1 & & & & & \\
 & & & & & & & & -1 & & 1 & & & & \\
 & & & & & & & & & -1 & & 1 & 1 & & \\
 & & & & & & & & & & -1 & -1 & & 1 & \\
 & & & & & & & & & & & & -1 & & 1 \\
 & & & & & & & & & & & & & -1 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15}
\end{bmatrix}
=
\begin{bmatrix}
-q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ q_{11} \\ -q_{12}
\end{bmatrix}
$$

Has it a solution?
Is the solution unique?

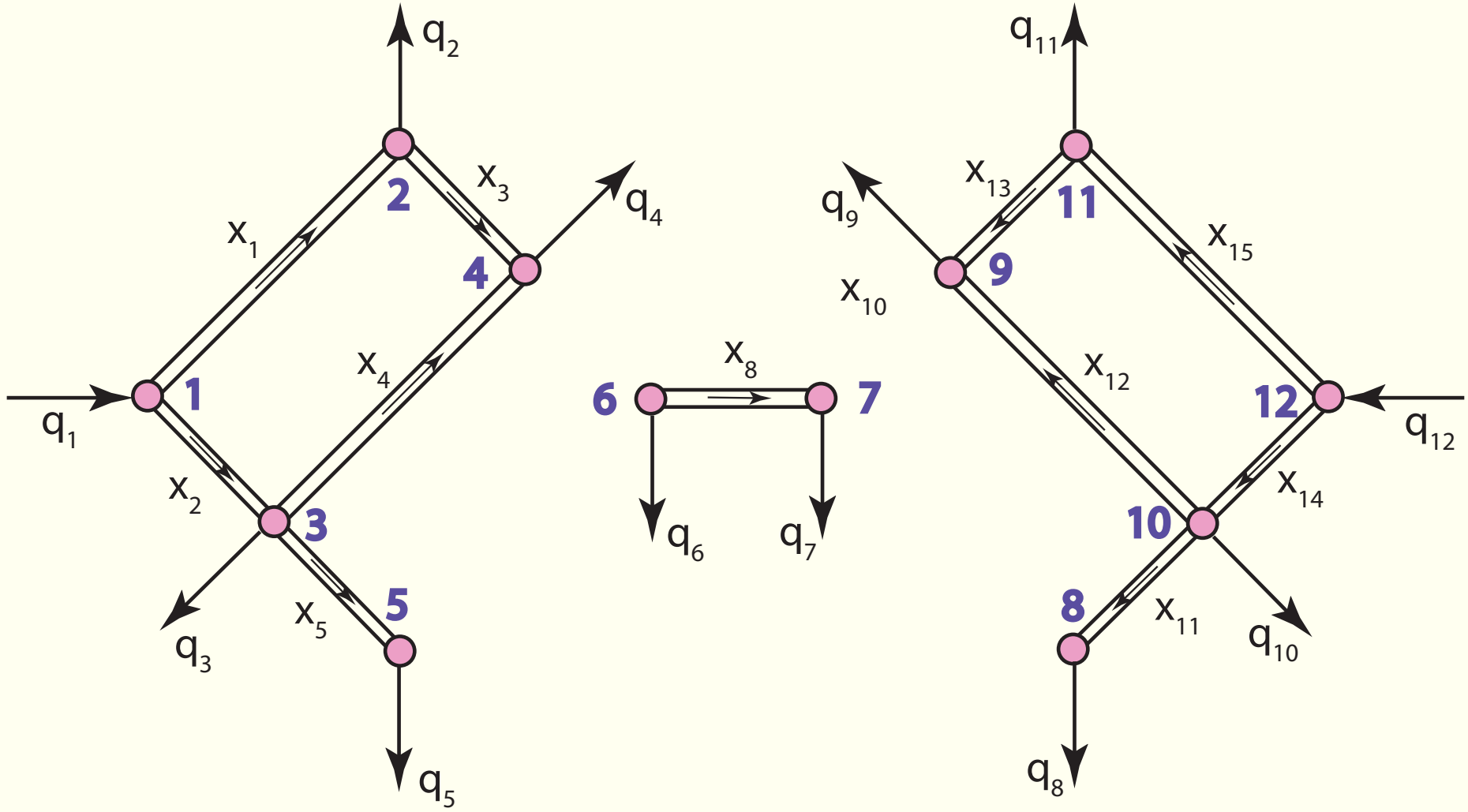
COMPATIBILITY SOLUTION

Incoming flow = outflow : $\sum_{i=1}^{12} q_i = 0$



COMPATIBILITY SOLUTION

$$
\begin{aligned}
-q_1 + q_2 + q_3 + q_4 + q_5 &= 0 \\
q_6 + q_7 &= 0 \\
q_8 + q_9 + q_{10} + q_{11} - q_{12} &= 0
\end{aligned}
$$



SET OF SOLUTIONS WITHOUT CAPACITY LIMITS

The student must determine the dimension of the linear space and obtain all solutions.

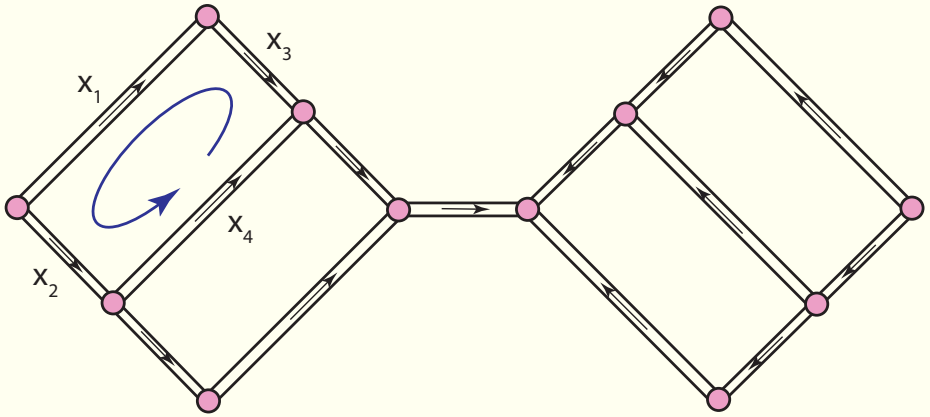
There are infinite solutions with four degrees of freedom because there are four holes in the network.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \rho_1 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \\ 5 \\ 0 \\ 2 \\ 2 \\ 0 \\ 1 \\ 6 \\ -4 \\ 10 \\ 0 \\ -3 \\ 12 \\ 0 \end{pmatrix}$$

The solution is an affine space associated with the linear space.

SET OF SOLUTIONS

Interpretation of the solutions: Circulating flow in the first hole

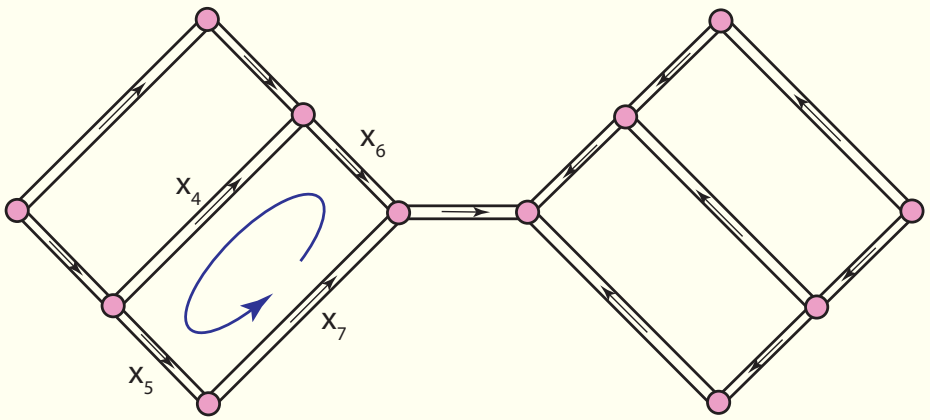


First basic solution of local internal flow without inputs and outputs of fluid corresponding to the associated homogeneous system

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \rho_1 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \\ 5 \\ 0 \\ 2 \\ 2 \\ 0 \\ 1 \\ 6 \\ -4 \\ 10 \\ 0 \\ -3 \\ 12 \\ 0 \end{pmatrix}$$

SET OF SOLUTIONS

Interpretation of the solutions: Circulating flow in the second hole

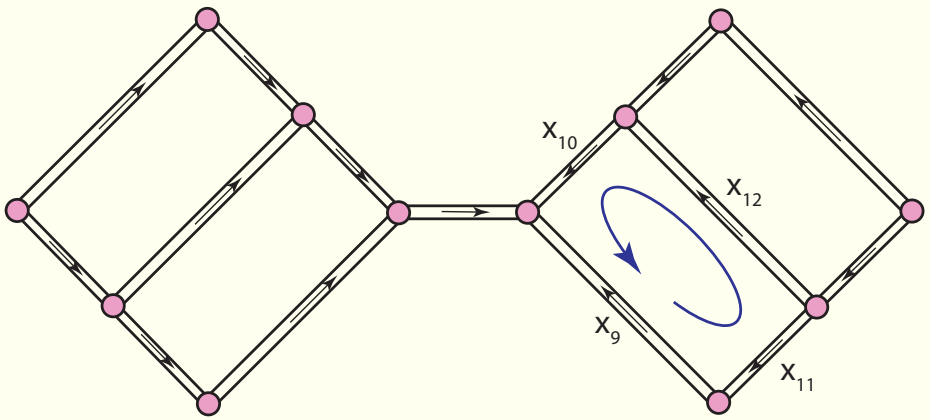


Second basic solution of local internal flow without inputs and outputs of fluid corresponding to the associated homogeneous system

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \rho_1 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \\ 5 \\ 0 \\ 2 \\ 2 \\ 0 \\ 1 \\ 6 \\ -4 \\ 10 \\ 0 \\ -3 \\ 12 \\ 0 \end{pmatrix}$$

SET OF SOLUTIONS

Interpretation of the solutions: Circulating flow in the third hole

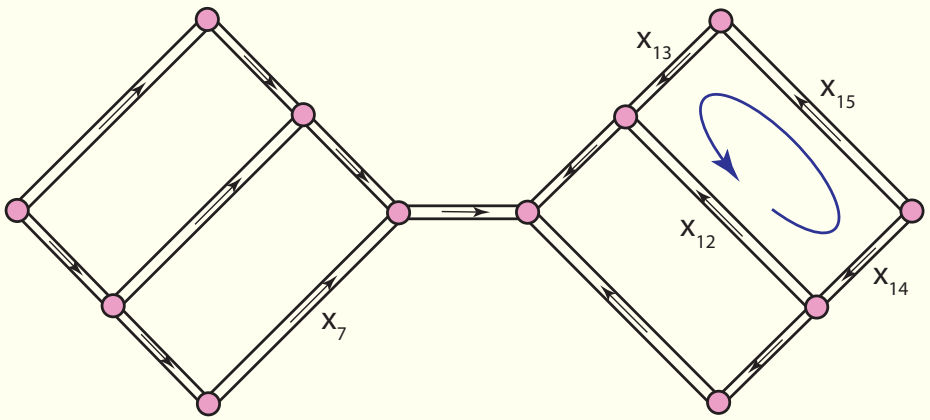


Third basic solution of local internal flow without inputs and outputs of fluid corresponding to the associated homogeneous system

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix}
= \rho_1 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
+ \rho_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
+ \rho_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
+ \rho_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}
+ \begin{pmatrix} 6 \\ 4 \\ 5 \\ 0 \\ 2 \\ 2 \\ 0 \\ 1 \\ 6 \\ -4 \\ 10 \\ 0 \\ -3 \\ 12 \\ 0 \end{pmatrix}
$$

SET OF SOLUTIONS

Interpretation of the solutions: Circulating flow in the fourth hole

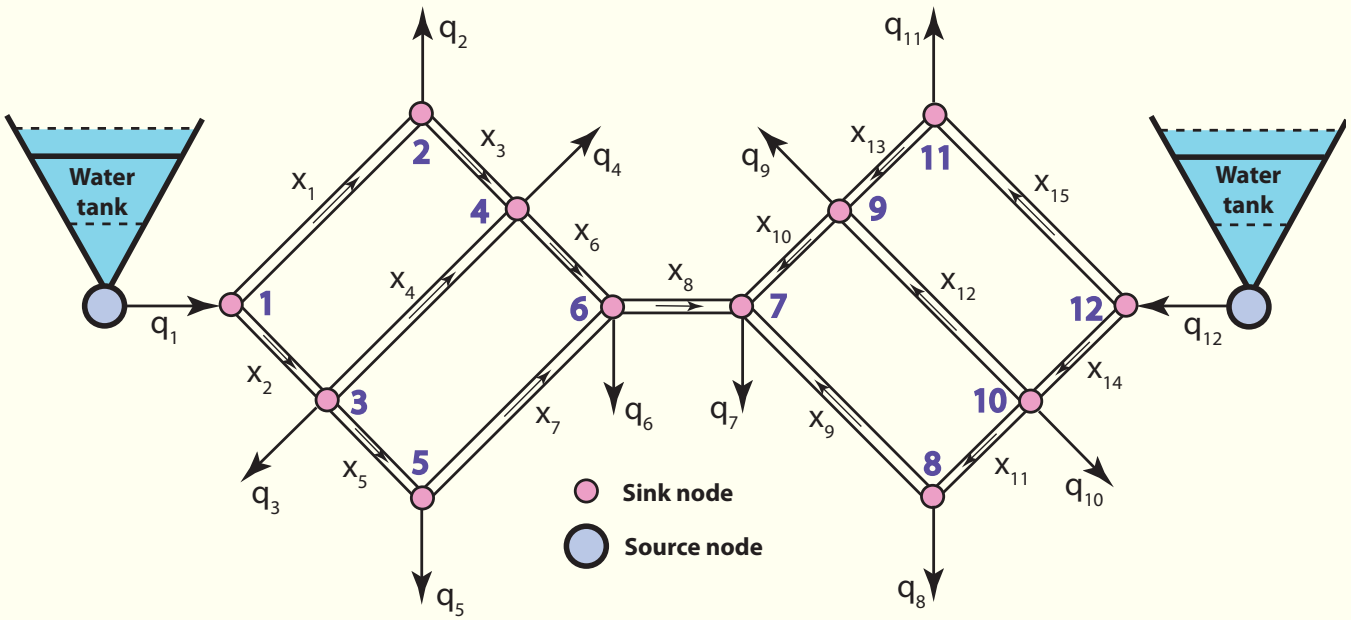


Fourth basic solution of local internal flow without inputs and outputs of fluid corresponding to the associated homogeneous system

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \rho_1 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \\ 5 \\ 0 \\ 2 \\ 2 \\ 0 \\ 1 \\ 6 \\ -4 \\ 10 \\ 0 \\ -3 \\ 12 \\ 0 \end{pmatrix}$$

SET OF SOLUTIONS

Interpretation of the solutions:
Particular solution of the complete system

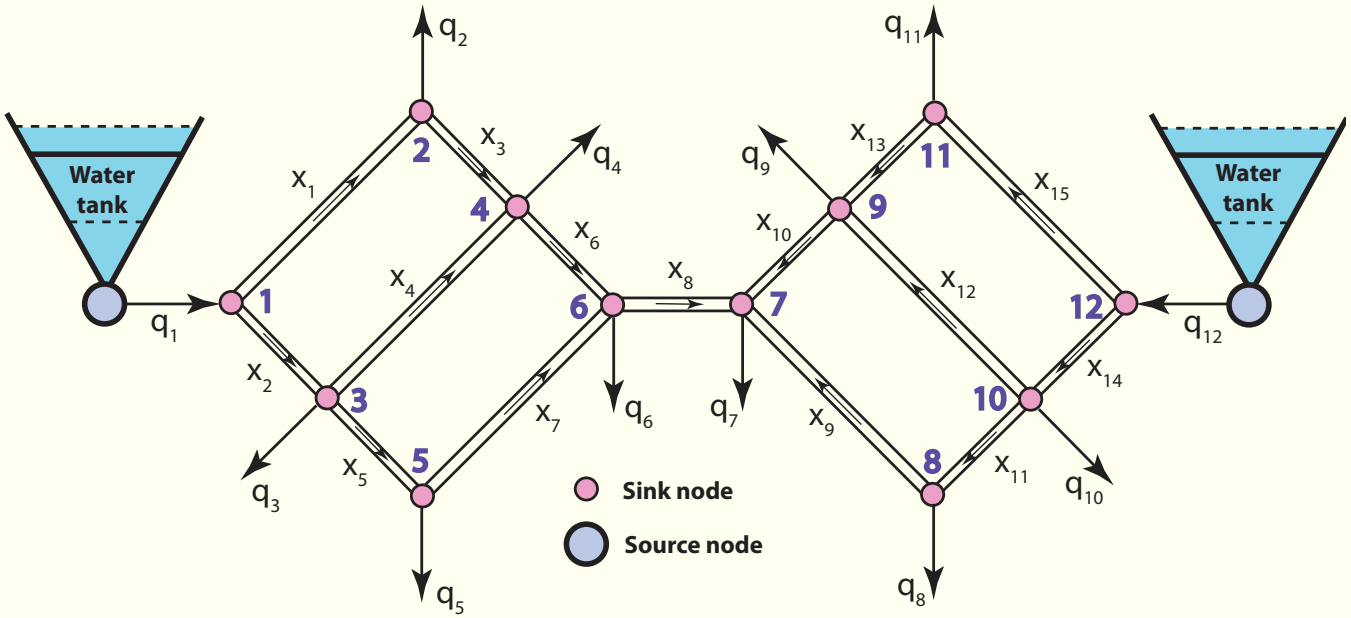


A particular solution that can be obtained by sending the flow to each destination by any possible path

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \rho_1 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \rho_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \\ 5 \\ 0 \\ 2 \\ 2 \\ 0 \\ 1 \\ 6 \\ -4 \\ 10 \\ 0 \\ -3 \\ 12 \\ 0 \end{pmatrix}$$

STATEMENT OF THE PROBLEM WITH CAPACITY LIMITS

Statement of the problem with all restrictions



$$
\begin{bmatrix}
-1 & -1 & & & & & & & & & & & & & \\
1 & & -1 & & & & & & & & & & & & \\
& 1 & & -1 & -1 & & & & & & & & & & \\
& & 1 & 1 & & -1 & & & & & & & & & \\
& & & & 1 & & -1 & & & & & & & & \\
& & & & & 1 & 1 & -1 & & & & & & & \\
& & & & & & & 1 & 1 & 1 & & & & & \\
& & & & & & & & -1 & & 1 & & & & \\
& & & & & & & & & -1 & & 1 & 1 & & \\
& & & & & & & & & & -1 & -1 & & 1 & \\
& & & & & & & & & & & & -1 & & 1 \\
& & & & & & & & & & & & & -1 & -1
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{bmatrix}
=
\begin{bmatrix} -q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ q_{11} \\ -q_{12} \end{bmatrix}
$$

$$-c_i \le x_i \le c_i; \quad i = 1, 2, \ldots, 15.$$

The student should identify inappropriate models and identify the missing restrictions, such as pipeline capacities.

STATEMENT OF THE PROBLEM WITH CAPACITY LIMITS

$$
(-1)^p \begin{pmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
-q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \\ q_9 \\ q_{10} \\ q_{11} \\ -q_{12}
\end{pmatrix}
\leq
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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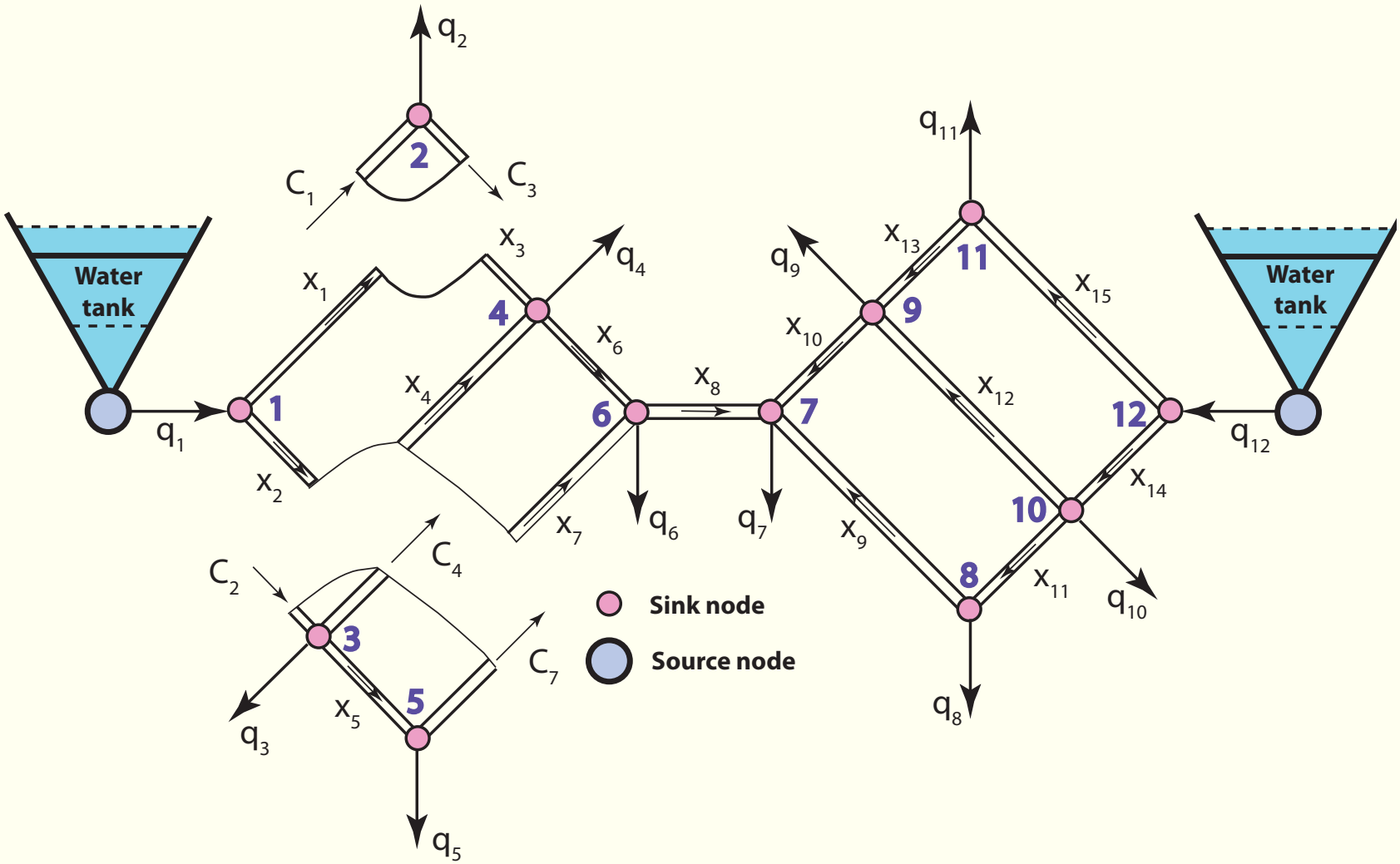
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0
\end{pmatrix}
\begin{pmatrix}
c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \\ c_{11} \\ c_{12} \\ c_{13} \\ c_{14} \\ c_{15}
\end{pmatrix}
$$

Compatibility conditions.

It is necessary to interpret them physically to see if they represent the desired model.

COMPATIBILITY CONDITIONS

$$-c_1 - c_3 \leq q_2 \leq c_1 + c_3$$



$$-c_2 - c_4 - c_7 \leq q_3 + q_5 \leq c_2 + c_4 + c_7$$

SET OF SOLUTIONS WITH CAPACITY LIMITS

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 4 & 6 & 4 & 6 & 4 & 6 & 4 & 6 \\ 6 & 4 & 6 & 4 & 6 & 4 & 6 & 4 \\ 3 & 5 & 3 & 5 & 3 & 5 & 3 & 5 \\ 6 & -4 & -2 & 4 & 6 & -4 & -2 & 4 \\ -2 & 6 & 6 & -2 & -2 & 6 & 6 & -2 \\ 6 & -2 & -2 & 6 & 6 & -2 & -2 & 6 \\ -4 & 4 & 4 & -4 & -4 & 4 & 4 & -4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -4 & -4 & -4 & -4 & 2 & 2 & 2 & 2 \\ 6 & 6 & 6 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 6 & 6 & 6 \\ 4 & 4 & 4 & 4 & -2 & -2 & -2 & -2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \end{pmatrix}; \qquad \sum_{i=1}^{8} \lambda_i = 1;\ \lambda_i \geq 0;\ i = 1, 2, \ldots, 8$$

Pipeline capacities = 6

The solution is a polytope.

The set of all solutions is useful to answer many interesting questions from the mathematical and engineering point of view.

SET OF SOLUTIONS WITH CAPACITY LIMITS

Seach for oversized pipelines

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 4 & 6 & 4 & 6 & 4 & 6 & 4 & 6 \\ 6 & 4 & 6 & 4 & 6 & 4 & 6 & 4 \\ 3 & 5 & 3 & 5 & 3 & 5 & 3 & 5 \\ 6 & -4 & -2 & 4 & 6 & -4 & -2 & 4 \\ -2 & 6 & 6 & -2 & -2 & 6 & 6 & -2 \\ 6 & -2 & -2 & 6 & 6 & -2 & -2 & 6 \\ -4 & 4 & 4 & -4 & -4 & 4 & 4 & -4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -4 & -4 & -4 & -4 & 2 & 2 & 2 & 2 \\ 6 & 6 & 6 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 6 & 6 & 6 \\ 4 & 4 & 4 & 4 & -2 & -2 & -2 & -2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \end{pmatrix}; \qquad \sum_{i=1}^{8} \lambda_i = 1;\ \lambda_i \geq 0;\ i = 1, 2, \ldots, 8$$

Pipeline capacities = 6

Pipelines 3,7,8,9,12 and 13 are oversized.

They are pipelines in which none of the components of the solutions reach their capacities. The capacity of each of them could be limited to that maximum without changing the solutions.

SET OF SOLUTIONS WITH CAPACITY LIMITS

Search for pipelines that cannot fail

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 4 & 6 & 4 & 6 & 4 & 6 & 4 & 6 \\ 6 & 4 & 6 & 4 & 6 & 4 & 6 & 4 \\ 3 & 5 & 3 & 5 & 3 & 5 & 3 & 5 \\ 6 & -4 & -2 & 4 & 6 & -4 & -2 & 4 \\ -2 & 6 & 6 & -2 & -2 & 6 & 6 & -2 \\ 6 & -2 & -2 & 6 & 6 & -2 & -2 & 6 \\ -4 & 4 & 4 & -4 & -4 & 4 & 4 & -4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -4 & -4 & -4 & -4 & 2 & 2 & 2 & 2 \\ 6 & 6 & 6 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 6 & 6 & 6 \\ 4 & 4 & 4 & 4 & -2 & -2 & -2 & -2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \end{pmatrix};$$

Pipeline capacities = 6

$$\sum_{i=1}^{8} \lambda_i = 1; \ \lambda_i \geq 0; \ i = 1, 2, \ldots, 8$$

Pipelines 1, 2, 3, 8, 13, 14 and 15 cannot fail without causing supply problems.

They take values of the same sign (all positive or all negative) in all solutions.

SET OF SOLUTIONS WITH CAPACITY LIMITS

Search for pipelines that cannot fail simultaneously

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 4 & 6 & 4 & 6 & 4 & 6 & 4 & 6 \\ 6 & 4 & 6 & 4 & 6 & 4 & 6 & 4 \\ 3 & 5 & 3 & 5 & 3 & 5 & 3 & 5 \\ 6 & -4 & -2 & 4 & 6 & -4 & -2 & 4 \\ -2 & 6 & 6 & -2 & -2 & 6 & 6 & -2 \\ 6 & -2 & -2 & 6 & 6 & -2 & -2 & 6 \\ -4 & 4 & 4 & -4 & -4 & 4 & 4 & -4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -4 & -4 & -4 & -4 & 2 & 2 & 2 & 2 \\ 6 & 6 & 6 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 6 & 6 & 6 \\ 4 & 4 & 4 & 4 & -2 & -2 & -2 & -2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \end{pmatrix};$$

Pipeline capacities = 6

$$\sum_{i=1}^{8} \lambda_i = 1; \ \lambda_i \geq 0; \ i = 1, 2, \ldots, 8$$

Pipelines 10 and 11 cannot fail simultaneously to maintain the water supply.

This condition implies that all lambdas must be null.

SET OF SOLUTIONS WITH CAPACITY LIMITS

Search for pipelines with fixed flow

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 4 & 6 & 4 & 6 & 4 & 6 & 4 & 6 \\ 6 & 4 & 6 & 4 & 6 & 4 & 6 & 4 \\ 3 & 5 & 3 & 5 & 3 & 5 & 3 & 5 \\ 6 & -4 & -2 & 4 & 6 & -4 & -2 & 4 \\ -2 & 6 & 6 & -2 & -2 & 6 & 6 & -2 \\ 6 & -2 & -2 & 6 & 6 & -2 & -2 & 6 \\ -4 & 4 & 4 & -4 & -4 & 4 & 4 & -4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -4 & -4 & -4 & -4 & 2 & 2 & 2 & 2 \\ 6 & 6 & 6 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 6 & 6 & 6 \\ 4 & 4 & 4 & 4 & -2 & -2 & -2 & -2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \end{pmatrix}; \qquad \sum_{i=1}^{8} \lambda_i = 1;\ \lambda_i \geq 0;\ i = 1, 2, \ldots, 8$$

Pipeline capacities = 6

Pipelines 8, 13, 14 and 15 always have the same flow.

The flow rates of the pipes 8, 13, 14 and 15 are always the same and equal to 1, 3, 6 and 6, respectively.

SET OF SOLUTIONS WITH CAPACITY LIMITS

Pipeline 10 damaged

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 4 & 6 & 4 & 6 & 4 & 6 & 4 & 6 \\ 6 & 4 & 6 & 4 & 6 & 4 & 6 & 4 \\ 3 & 5 & 3 & 5 & 3 & 5 & 3 & 5 \\ 6 & -4 & -2 & 4 & 6 & -4 & -2 & 4 \\ -2 & 6 & 6 & -2 & -2 & 6 & 6 & -2 \\ 6 & -2 & -2 & 6 & 6 & -2 & -2 & 6 \\ -4 & 4 & 4 & -4 & -4 & 4 & 4 & -4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -4 & -4 & -4 & -4 & 2 & 2 & 2 & 2 \\ 6 & 6 & 6 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 6 & 6 & 6 \\ 4 & 4 & 4 & 4 & -2 & -2 & -2 & -2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \\ \lambda_6 \\ \lambda_7 \\ \lambda_8 \end{pmatrix};$$

Pipeline capacities = 6

$$\sum_{i=1}^{8} \lambda_i = 1; \ \lambda_i \geq 0; \ i = 1, 2, \ldots, 8$$

For pipeline 10 to have zero flow, the the first four lambdas must be null.

Therefore the remaining solutions are given by the other columns.

SET OF SOLUTIONS WITH CAPACITY LIMITS

Pipeline 10 damaged

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 4 & 6 & 4 & 6 \\ 6 & 4 & 6 & 4 \\ 3 & 5 & 3 & 5 \\ 6 & -4 & -2 & 4 \\ -2 & 6 & 6 & -2 \\ 6 & -2 & -2 & 6 \\ -4 & 4 & 4 & -4 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 6 & 6 & 6 & 6 \\ -2 & -2 & -2 & -2 \\ 3 & 3 & 3 & 3 \\ 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix};$$

Pipeline capacities = 6

$$\sum_{i=1}^{4} \lambda_i = 1; \ \lambda_i \geq 0; \ i = 1, 2, 3, 4$$

Can pipeline 7 fail?

The pipeline 7 can fail because it has positive and negative components.

If pipelines 7 and 10 fail, the supply is maintained.

SET OF SOLUTIONS WITH CAPACITY LIMITS

Pipeline 10 damaged

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 4 & 6 \\ 5 & 3 \\ 0 & 2 \\ 2 & 2 \\ 2 & 2 \\ 0 & 0 \\ 1 & 1 \\ 2 & 2 \\ 0 & 0 \\ 6 & 6 \\ -2 & -2 \\ 3 & 3 \\ 6 & 6 \\ 6 & 6 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix};$$

Pipeline capacities = 6

$$\sum_{i=1}^{2} \lambda_i = 1; \ \lambda_i \geq 0; \ i = 1, 2$$

Can pipeline 4 fail?

Pipeline 4 can fail if λ_2 is null.

If pipelines 4, 7 and 10 fail, the only solution is given by the first column of the matrix.

SET OF SOLUTIONS WITH CAPACITY LIMITS

Pipelines 4, 7 and 10 damaged

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 5 \\ 0 \\ 2 \\ 2 \\ 0 \\ 1 \\ 2 \\ 0 \\ 6 \\ -2 \\ 3 \\ 6 \\ 6 \end{pmatrix}$$

can any other pipeline fail?

None can fail, because the solution is unique.

The unique solution is bad engineering, because there is no flexibility left. It is not convenient because its vulnerability to failures.

SET OF SOLUTIONS WITH CAPACITY LIMITS

RETENTION VALVES IN PIPELINES 2 AND 15

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} + \begin{pmatrix} 10 \\ 0 \\ 9 \\ -4 \\ 2 \\ 2 \\ 0 \\ 1 \\ 0 \\ -4 \\ 10 \\ 0 \\ -3 \\ 12 \\ 0 \end{pmatrix}$$

The solution is a polyhedron, that is, the sum of a linear space plus a cone plus a polytope (a vector).