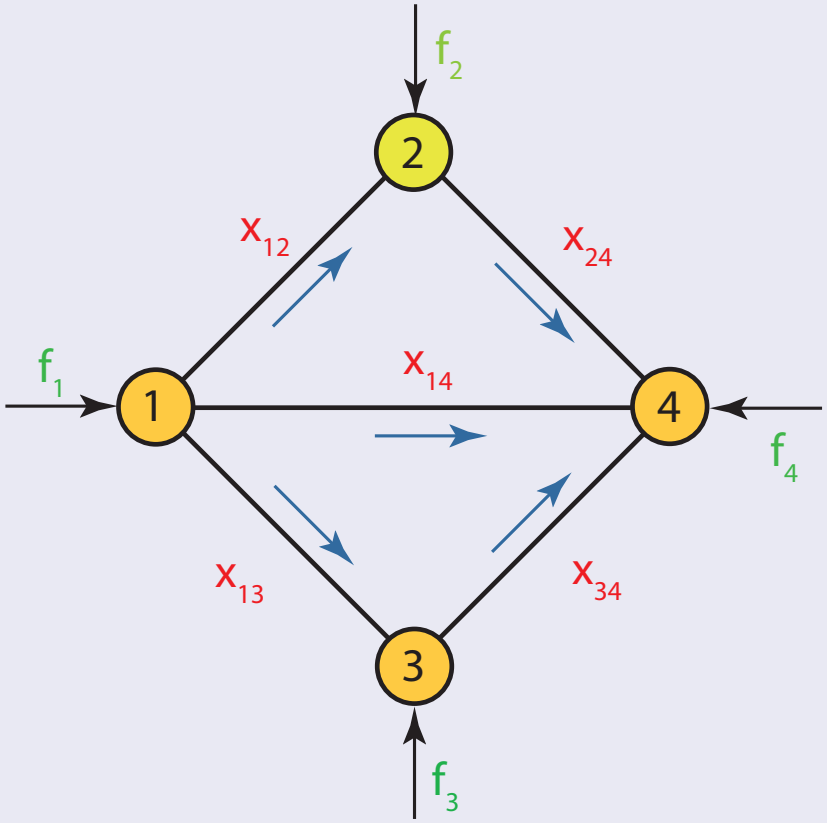
EXAMPLE 1

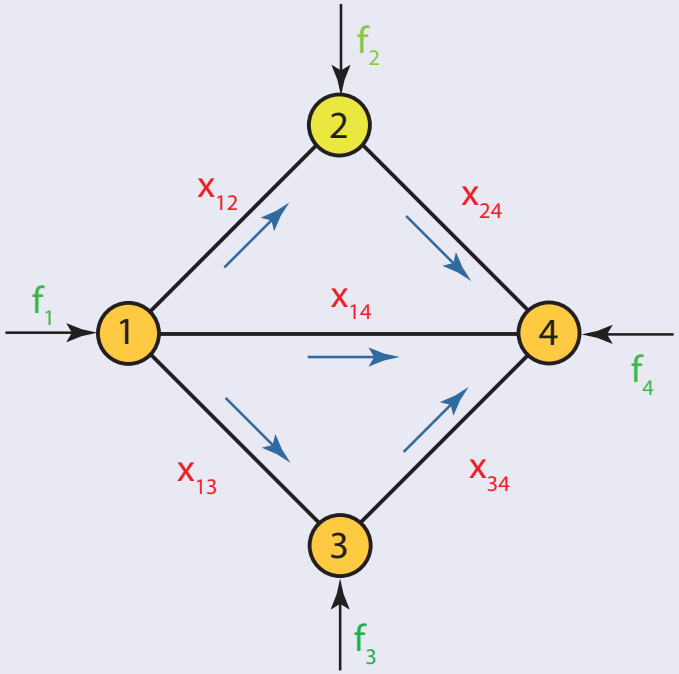
WATER SUPPLY NETWORK PROBLEM

Consider the following water supply network:



where f_1, f_2, f_3, f_4 are the income and outcome flows, and $x_{12}, x_{13}, x_{14}, x_{24}, x_{34}$ are the link flows.

WATER SUPPLY NETWORK PROBLEM



The flow balance at the nodes provides the following system of linear equations:

$$
\begin{array}{lrrrrrcl}
Node\ 1: & x_{12} & +x_{13} & +x_{14} & & & = & f_1 \\
Node\ 2: & -x_{12} & & & +x_{24} & & = & f_2 \\
Node\ 3: & & -x_{13} & & & +x_{34} & = & f_3 \\
Node\ 4: & & & -x_{14} & -x_{24} & -x_{34} & = & f_4
\end{array}
$$

First, we study the **compatibility conditions** for this system to have a solution.

COMPATIBILITY

$$
\begin{array}{lrrrrrcl}
Node\ 1: & x_{12} & +x_{13} & +x_{14} & & & = & f_1 \\
Node\ 2: & -x_{12} & & & +x_{24} & & = & f_2 \\
Node\ 3: & & -x_{13} & & & +x_{34} & = & f_3 \\
Node\ 4: & & & -x_{14} & -x_{24} & -x_{34} & = & f_4
\end{array}
$$

Iteration 1					Iteration 2					Iteration 3					Iteration 4					Iteration 5					Final Tableau			
a_1	v_1^1	v_2^1	v_3^1	v_4^1	a_2	w_1^2	v_1^2	v_2^2	v_3^2	a_3	w_1^3	w_2^3	v_1^3	v_2^3	a_4	w_1^4	w_2^4	w_3^4	v_1^4	a_5	w_1^5	w_2^5	w_3^5	v_1^5	w_1	w_2	w_3	v_1
1	**1**	0	0	0	1	1	**1**	0	0	1	0	1	**1**	0	0	0	0	1	1	0	0	0	1	1	0	0	1	1
-1	**0**	1	0	0	0	0	**1**	0	0	0	-1	1	**1**	0	1	-1	0	1	1	0	-1	0	1	1	-1	0	1	1
0	**0**	0	1	0	-1	0	**0**	1	0	0	0	0	**1**	0	0	0	-1	1	1	1	0	-1	1	1	0	-1	1	1
0	**0**	0	0	1	0	0	**0**	0	1	-1	0	0	**0**	1	-1	0	0	0	1	-1	0	0	0	1	0	0	0	1
t^1	**1**	-1	0	0	t^2	1	**1**	-1	0	t^3	0	1	**1**	-1	t^4	-1	0	1	0	t^5	0	-1	1	0				

Removing the pivot columns we get only a single column, and then **the compatibility condition becomes**:

$$(1,1,1,1) \circ (f_1, f_2, f_3, f_4) = f_1 + f_2 + f_3 + f_4 = 0,$$

which implies that **the total income and outcome flow must be null**.

SOLUTION

$$
\begin{array}{lrrrrrcr}
Node\ 1: & x_{12} & +x_{13} & +x_{14} & & & = & 3 \\
Node\ 2: & -x_{12} & & & +x_{24} & & = & -1 \\
Node\ 3: & & -x_{13} & & & +x_{34} & = & -1 \\
Node\ 4: & & & -x_{14} & -x_{24} & -x_{34} & = & -1
\end{array}
$$

The solution of this system can be obtained as follows:

Iteration 1						
a_1	v_1^1	v_2^1	v_3^1	v_4^1	v_5^1	v_6^1
1	**1**	0	0	0	0	0
1	**0**	1	0	0	0	0
1	**0**	0	1	0	0	0
0	**0**	0	0	1	0	0
0	**0**	0	0	0	1	0
-3	**0**	0	0	0	0	1
t^1	**1**	1	1	0	0	-3

Iteration 2						
a_2	w_1^2	v_1^2	v_2^2	v_3^2	v_4^2	v_5^2
-1	1	**-1**	-1	0	0	3
0	0	**1**	0	0	0	0
0	0	**0**	1	0	0	0
1	0	**0**	0	1	0	0
0	0	**0**	0	0	1	0
1	0	**0**	0	0	0	1
t^2	-1	**1**	1	1	0	-2

Iteration 3						
a_3	w_1^3	w_2^3	v_1^3	v_2^3	v_3^3	v_4^3
0	0	-1	**0**	1	0	1
-1	1	1	**-1**	-1	0	2
0	0	0	**1**	0	0	0
0	0	0	**0**	1	0	0
1	0	0	**0**	0	1	0
1	0	0	**0**	0	0	1
t^3	-1	-1	**1**	1	1	-1

Iteration 4						
a_4	w_1^4	w_2^4	w_3^4	v_1^4	v_2^4	v_3^4
0	0	-1	0	1	0	1
0	0	0	-1	0	1	1
-1	1	1	1	-1	-1	1
-1	0	0	0	1	0	0
-1	0	0	0	0	1	0
1	0	0	0	0	0	1
t^4	-1	-1	-1	0	0	0

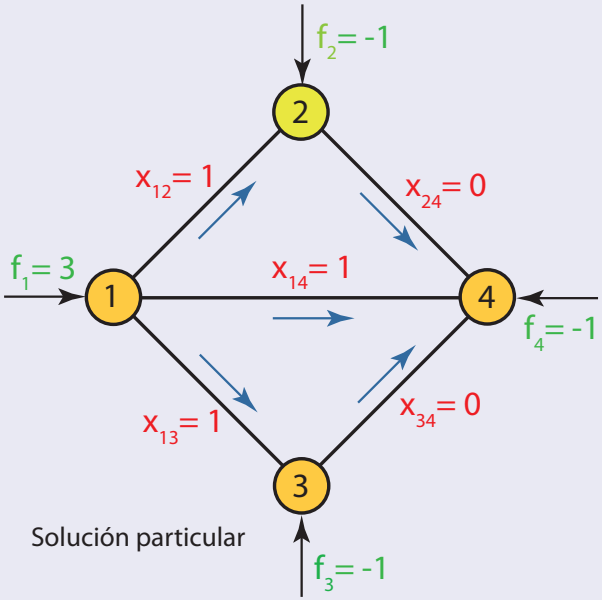
Final Tableau					
w_1	w_2	w_3	v_1	v_2	v_3
0	-1	0	1	0	1
0	0	-1	0	1	1
1	1	1	-1	-1	1
0	0	0	1	0	0
0	0	0	0	1	0
0	0	0	0	0	1

Removing the pivot columns we get three columns, that leads to the following solution:

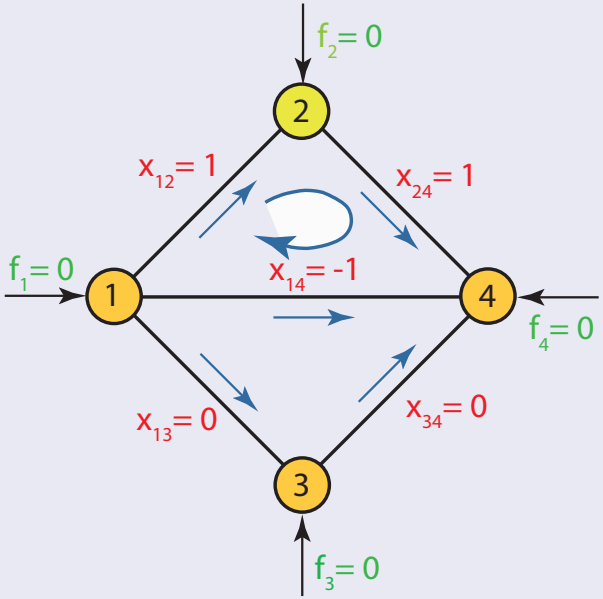
$$
\begin{pmatrix} x_{12} \\ x_{13} \\ x_{14} \\ x_{24} \\ x_{34} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \rho_1 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \rho_2 \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.
$$

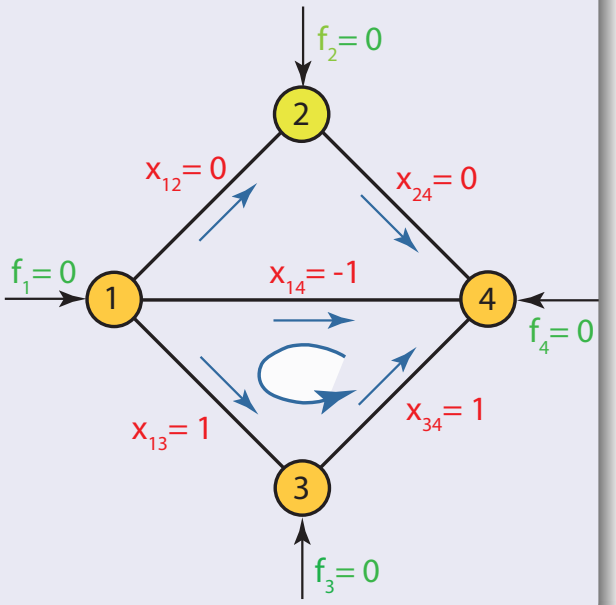
Water supply network problem

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$



Solución particular

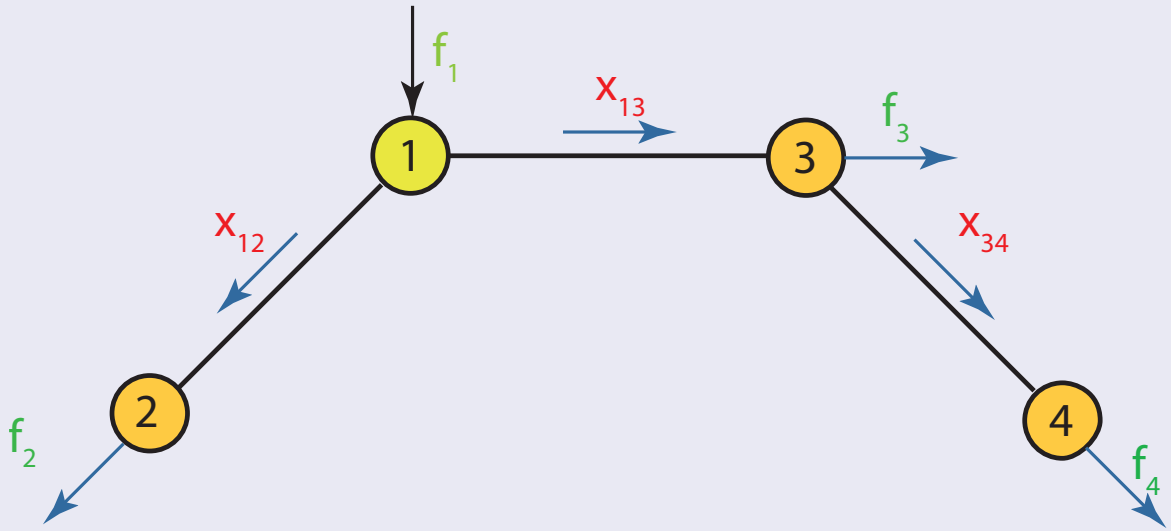




EXAMPLE 2

WATER SUPPLY NETWORK PROBLEM

Consider now the water supply network:



where f_1, f_2, f_3 are the income and outcome flows, and x_{12}, x_{13}, x_{34} are the link flows.

The flow balance at the nodes provides the following system of linear equations:

$$
\begin{array}{lrrrcl}
Node\ 1: & x_{12} & +x_{13} & & = & f_1 \\
Node\ 2: & -x_{12} & & & = & -f_2 \\
Node\ 3: & & -x_{13} & +x_{34} & = & -f_3 \\
Node\ 4: & & & -x_{34} & = & -f_4
\end{array}
$$

Compatibility

$$
\begin{array}{lrrrrcl}
Node\ 1: & x_{12} & +x_{13} & & & = & f_1 \\
Node\ 2: & -x_{12} & & & & = & -f_2 \\
Node\ 3: & & -x_{13} & & +x_{34} & = & -f_3 \\
Node\ 4: & & & & -x_{34} & = & -f_4
\end{array}
$$

Iteration 1				
a_1	v_1^1	v_2^1	v_3^1	v_4^1
1	**1**	0	0	0
-1	**0**	1	0	0
0	**0**	0	1	0
0	**0**	0	0	1
t^1	**1**	-1	0	0

Iteration 2				
a_2	w_1^2	v_1^2	v_2^2	v_3^2
1	1	**1**	0	0
0	0	**1**	0	0
-1	0	**0**	1	0
0	0	**0**	0	1
t^2	1	**1**	-1	0

Iteration 3				
a_3	w_1^3	w_2^3	v_1^3	v_2^3
0	0	1	**1**	0
0	-1	1	**1**	0
1	0	0	**1**	0
-1	0	0	**0**	1
t^3	0	0	**1**	-1

Final Tableau			
w_1	w_2	w_3	v_1
0	1	1	**1**
-1	1	1	**1**
0	0	1	**1**
0	0	0	**1**

Removing the pivot columns we get only a single column, and then **the compatibility condition becomes**:

$$(1,1,1,1) \circ (f_1, -f_2, -f_3, -f_4) = f_1 - f_2 - f_3 - f_4 = 0,$$

which implies that **the total income and outcome flow must be null**.

SOLUTION

$$
\begin{array}{lrrrcl}
Node\ 1: & x_{12} & +x_{13} & & = & 3 \\
Node\ 2: & -x_{12} & & & = & -1 \\
Node\ 3: & & -x_{13} & +x_{34} & = & -1 \\
Node\ 4: & & & -x_{34} & = & -1
\end{array}
$$

The solution of this system can be obtained as follows:

Iteration 1				
a_1	v_1^1	v_2^1	v_3^1	v_4^1
1	**1**	0	0	0
1	**0**	1	0	0
0	**0**	0	1	0
-3	**0**	0	0	1
t^1	**1**	1	0	-3

Iteration 2				
a_2	w_1^2	v_1^2	v_2^2	v_3^2
-1	1	**-1**	0	3
0	0	**1**	0	0
0	0	**0**	1	0
1	0	**0**	0	1
t^2	-1	**1**	0	-2

Iteration 3				
a_3	w_1^3	w_2^3	v_1^3	v_2^3
0	0	-1	**0**	1
-1	1	1	**0**	2
1	0	0	**1**	0
1	0	0	**0**	1
t^3	-1	-1	**1**	-1

Iteration 4				
a_4	w_1^4	w_2^4	w_3^4	v_1^4
0	0	-1	0	1
0	1	1	0	2
-1	1	1	1	1
1	0	0	0	1
t^4	-1	-1	-1	0

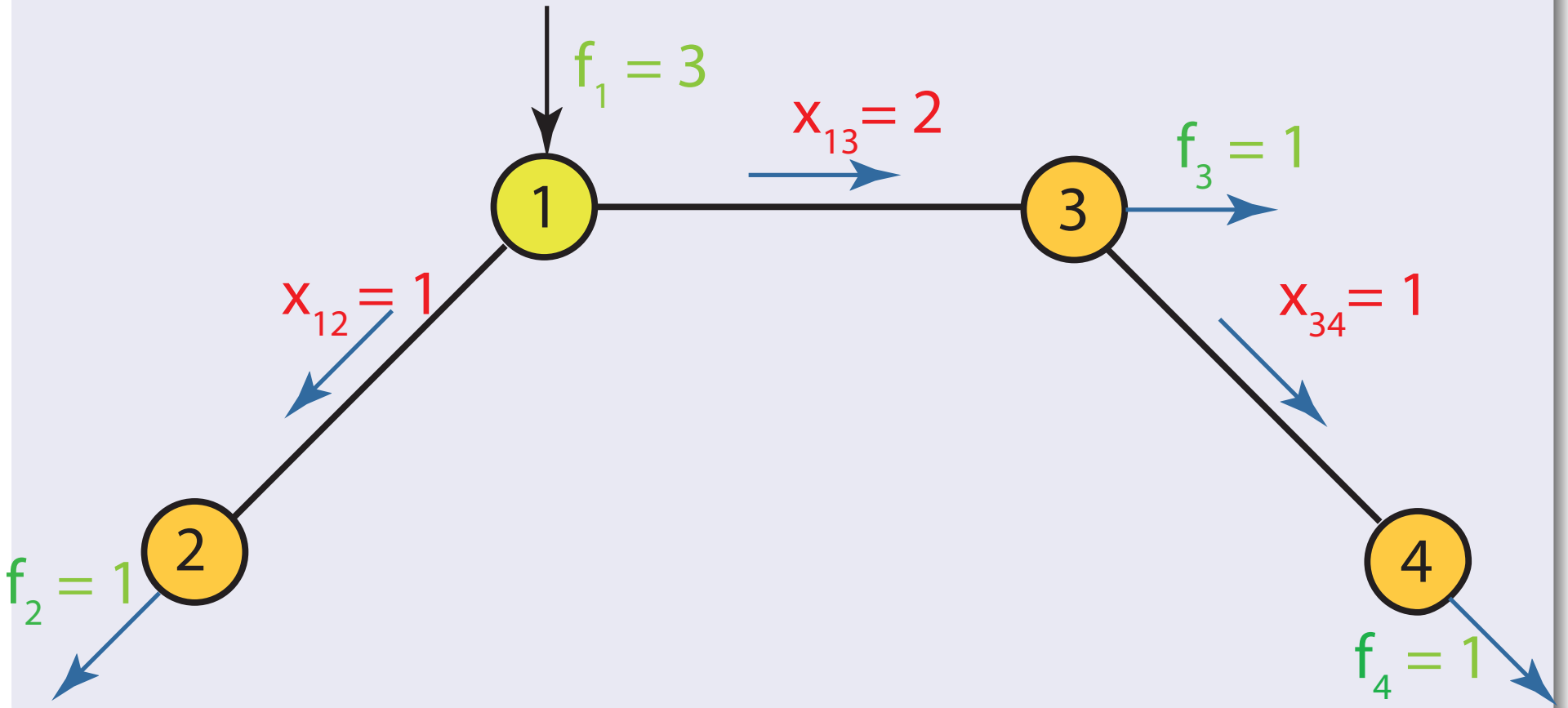
Final Tableau			
w_1	w_2	w_3	v_1
0	-1	0	1
1	1	0	2
1	1	1	1
0	0	0	1

Removing the pivot columns we get three columns, that leads to the following unique solution:

$$
\begin{pmatrix} x_{12} \\ x_{13} \\ x_{34} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.
$$

Water supply network problem

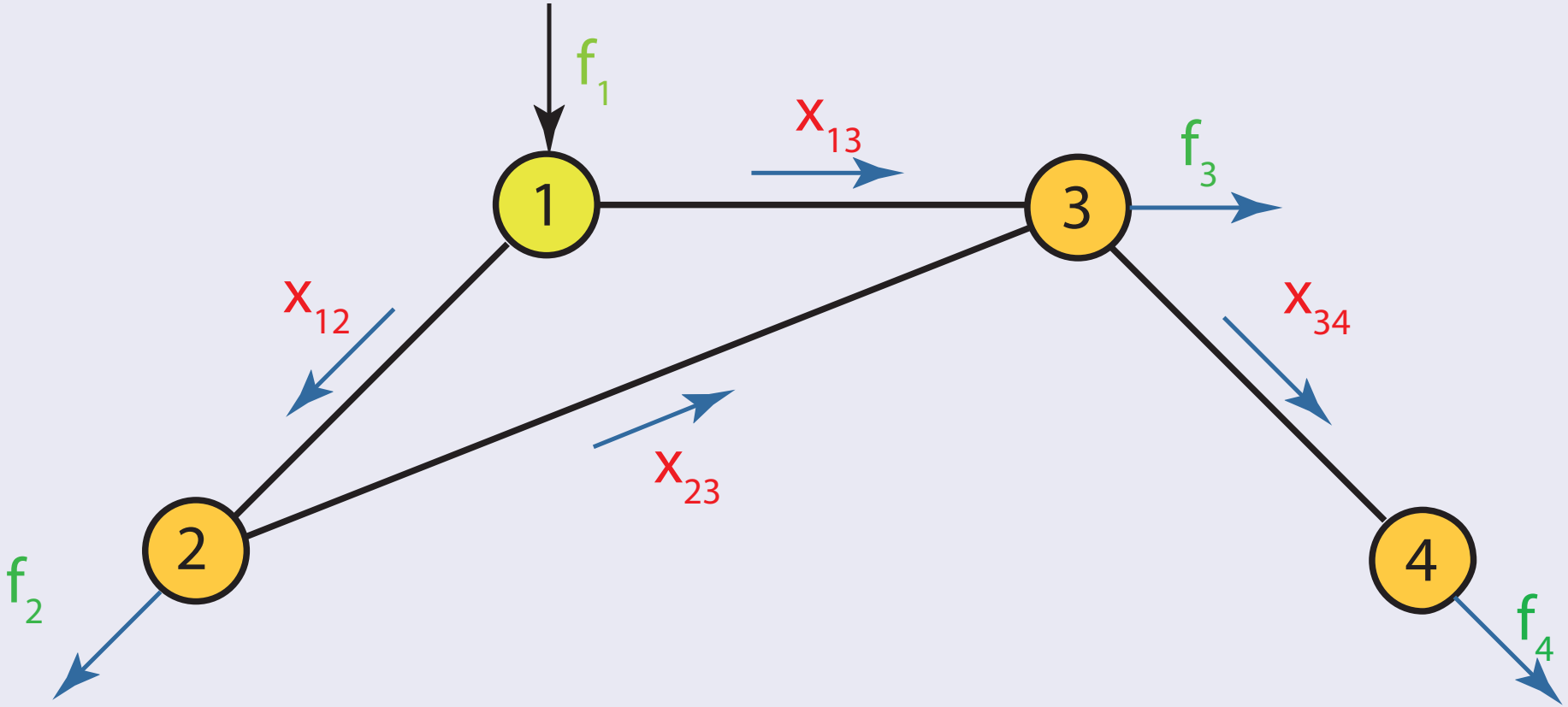
$$\begin{pmatrix} x_{12} \\ x_{13} \\ x_{34} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$



EXAMPLE 3

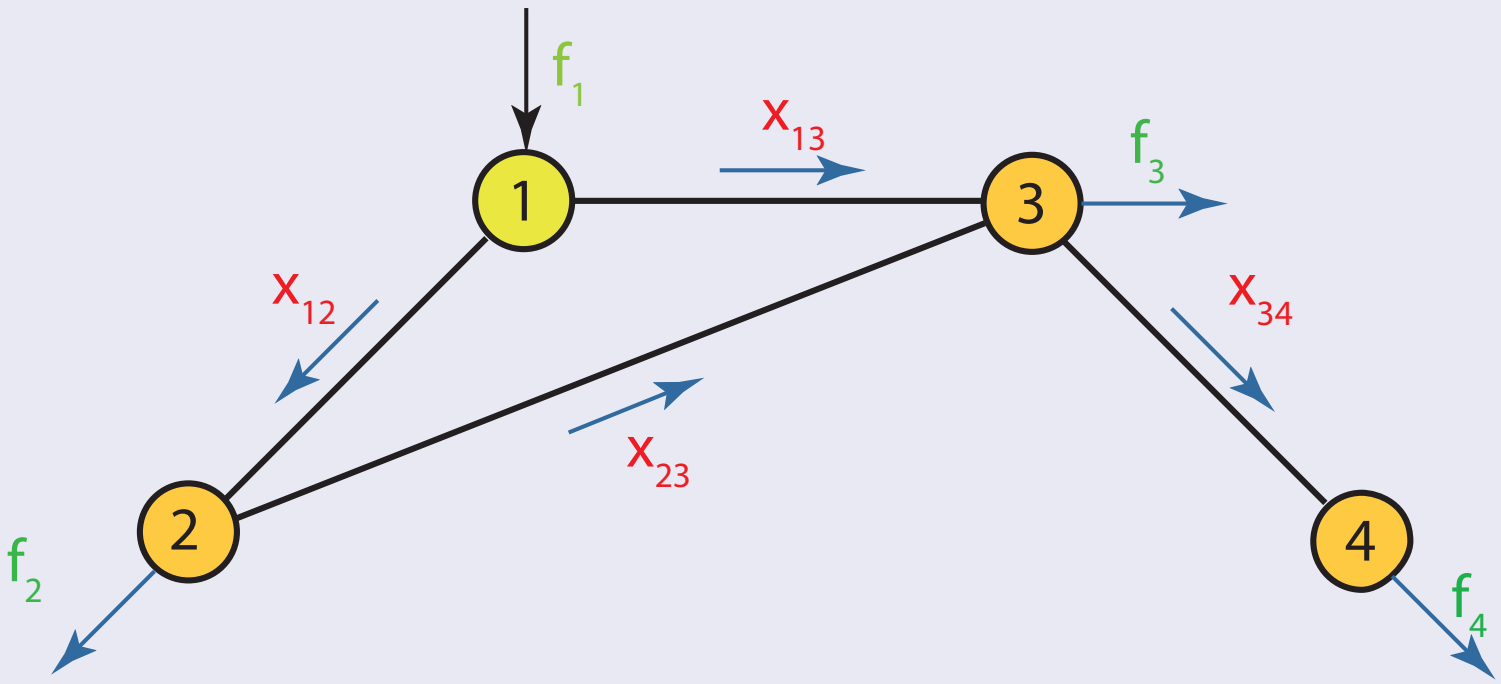
WATER SUPPLY NETWORK PROBLEM

Finally, consider the water supply network:



where f_1, f_2, f_3, f_4 are the income and outcome flows, and $x_{12}, x_{13}, x_{23}, x_{34}$ are the link flows.

WATER SUPPLY NETWORK PROBLEM



The flow balance at the nodes provides the following system of linear equations:

$$
\begin{array}{lrrrrcl}
\textit{Node 1:} & x_{12} & +x_{13} & & & = & f_1 \\
\textit{Node 2:} & -x_{12} & & +x_{23} & & = & -f_2 \\
\textit{Node 3:} & & -x_{13} & +x_{34} & -x_{23} & = & -f_3 \\
\textit{Node 4:} & & & -x_{34} & & = & -f_4
\end{array}
$$

First, we study the **compatibility conditions** for this system to have a solution.

Compatibility

$$
\begin{array}{lrrrrrcl}
Node\ 1: & x_{12} & +x_{13} & & & & = & f_1 \\
Node\ 2: & -x_{12} & & & +x_{23} & & = & -f_2 \\
Node\ 3: & & -x_{13} & & +x_{34} & -x_{23} & = & -f_3 \\
Node\ 4: & & & & -x_{34} & & = & -f_4
\end{array}
$$

Iteration 1					Iteration 2					Iteration 3					Iteration 4					Final Tableau			
a_1	v_1^1	v_2^1	v_3^1	v_4^1	a_2	w_1^2	v_1^2	v_2^2	v_3^2	a_3	w_1^3	w_2^3	v_1^3	v_2^3	a_4	w_1^4	w_2^4	w_3^4	v_1^4	w_1	w_2	w_3	v_1
1	**1**	0	0	0	1	1	**1**	0	0	0	0	1	**1**	0	0	0	1	1	1	0	1	1	**1**
-1	**0**	1	0	0	0	0	**1**	0	0	0	-1	1	**1**	0	1	-1	1	1	1	-1	1	1	**1**
0	**0**	0	1	0	-1	0	**0**	1	0	1	0	0	**1**	0	-1	0	0	1	1	0	0	1	**1**
0	**0**	0	0	1	0	0	**0**	0	1	-1	0	0	**0**	1	0	0	0	0	1	0	0	0	**1**
t^1	**1**	-1	0	0	t^2	1	**1**	-1	0	t^3	0	0	**1**	-1	t^4	-1	1	0	0				

Removing the pivot columns we get only a single column, and then **the compatibility condition becomes**:

$$(1,1,1,1) \circ (f_1, -f_2, -f_3, -f_4) = f_1 - f_2 - f_3 - f_4 = 0,$$

which implies that **the total income and outcome flow must be null**.

SOLUTION

$$
\begin{array}{lrrrrrl}
Node\ 1: & x_{12} & +x_{13} & & & = & 3 \\
Node\ 2: & -x_{12} & & +x_{23} & & = & -1 \\
Node\ 3: & & -x_{13} & +x_{34} & -x_{23} & = & -1 \\
Node\ 4: & & & -x_{34} & & = & -1
\end{array}
$$

The solution of this system can be obtained as follows:

Iteration 1					
a_1	v_1^1	v_2^1	v_3^1	v_4^1	v_5^1
1	**1**	0	0	0	0
1	**0**	1	0	0	0
0	**0**	0	1	0	0
0	**0**	0	0	1	0
-3	**0**	0	0	0	1
t^1	**1**	1	0	0	-3

Iteration 2					
a_2	w_1^2	v_1^2	v_2^2	v_3^2	v_4^2
-1	1	**-1**	0	0	3
0	0	**1**	0	0	0
0	0	**0**	1	0	0
1	0	**0**	0	1	0
1	0	**0**	0	0	1
t^2	-1	**1**	0	1	-2

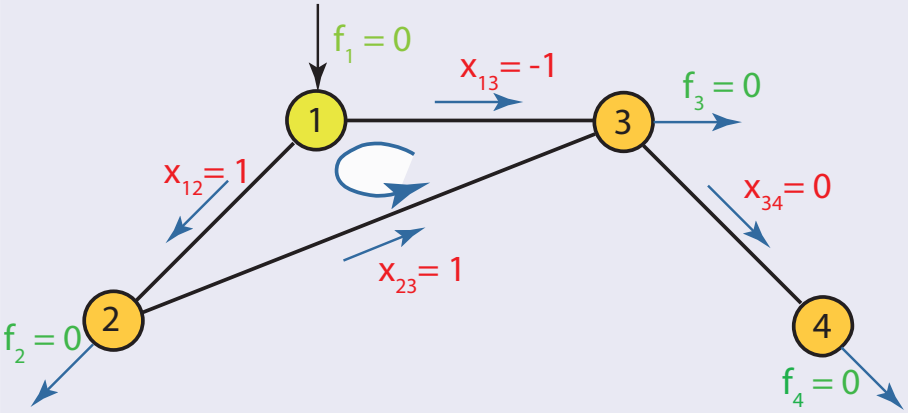
Iteration 3					
a_3	w_1^3	w_2^3	v_1^3	v_2^3	v_3^3
0	0	-1	**0**	1	1
-1	1	1	**0**	-1	2
1	0	0	**1**	0	0
-1	0	0	**0**	1	0
1	0	0	**0**	0	1
t^3	-1	-1	**1**	0	-1

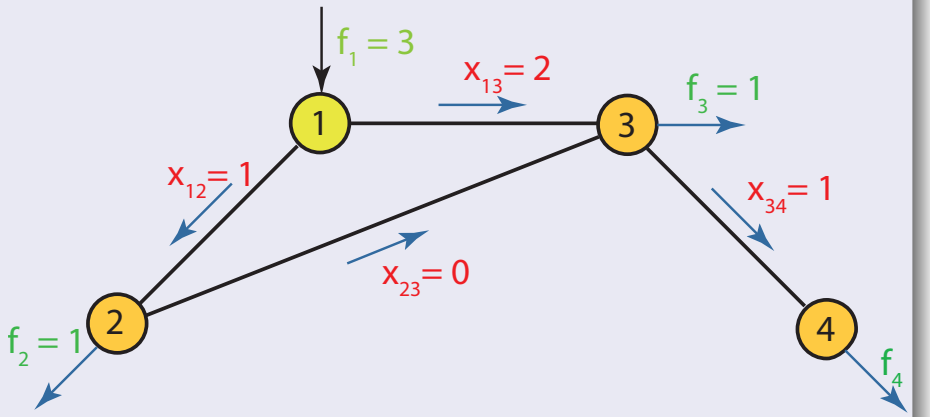
Iteration 4					
a_4	w_1^4	w_2^4	w_3^4	v_1^4	v_2^4
0	0	-1	0	1	1
0	1	1	0	-1	2
-1	1	1	1	0	1
0	0	0	0	1	0
1	0	0	0	0	1
t^4	-1	-1	-1	0	0

Final Tableau				
w_1	w_2	w_3	v_1	v_2
0	-1	0	**1**	**1**
1	1	0	**-1**	**2**
1	1	1	**0**	**1**
0	0	0	**1**	**0**
0	0	0	**0**	**1**

Water supply network problem

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}$$





$$\begin{pmatrix} x_{12} \\ x_{13} \\ x_{34} \\ x_{23} \end{pmatrix} = \rho \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$