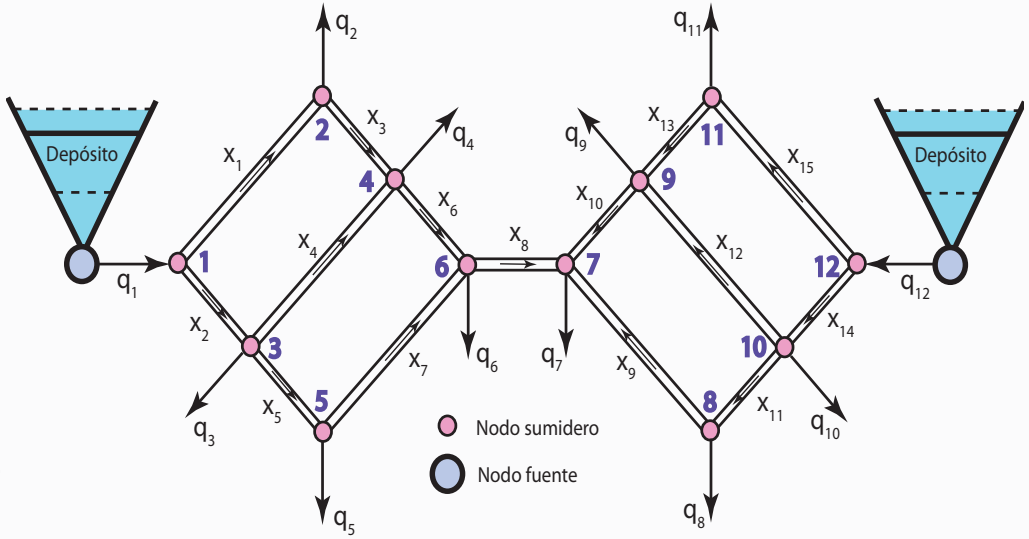
Algebra I

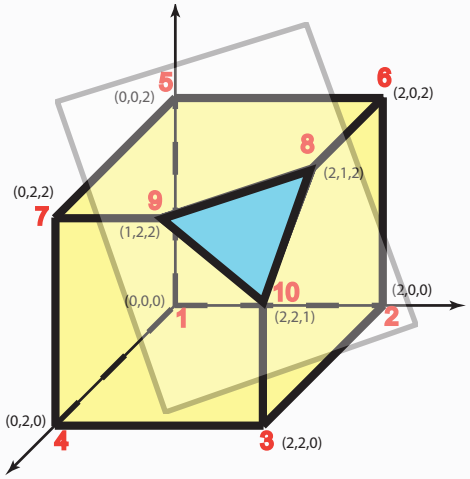
Enrique Castillo Ron
Universidad de Castilla-La Mancha

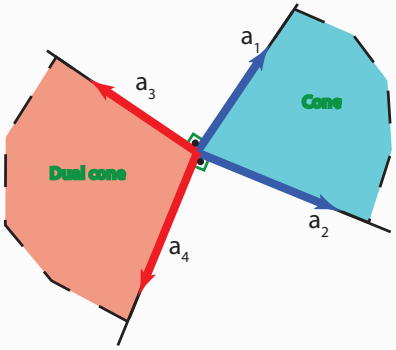
Royal Academy of Engineering

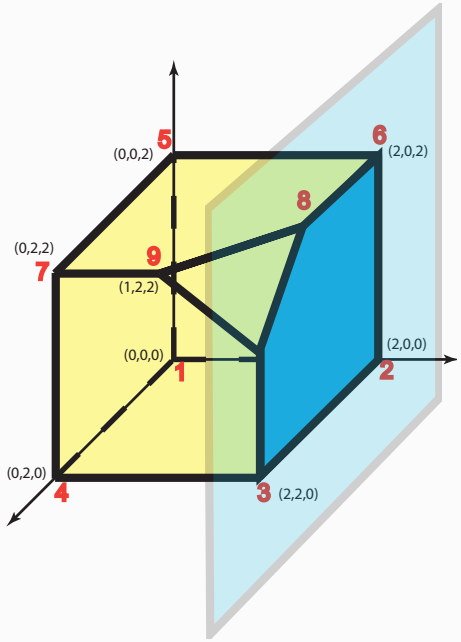
Royal Academy of Sciences

Enrique Castillo









Elemental transformations of matrices

Algebra

Matrices of type I

Type I matrix comes from the unit matrix changing a diagonal element by $q \neq 0$.

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix},$$

Postmultiplying matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

by the previous matrix, whose determinant is $|Q_1| = q$, we get

$$A\ Q_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & qa_{13} \\ a_{21} & a_{22} & qa_{23} \\ a_{31} & a_{32} & qa_{33} \end{pmatrix},$$

which is matrix A with its last column multiplied by q.

To multiply a column of a given matrix by a value it is enough to postmultiply it by a type I matrix.

Matrices of type II

A matrix of type II comes from the unit matrix changing a null element by $s \neq 0$.

$$Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix},$$

Postmultiplying the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

by the previous matrix, whose determinat is $|Q_2| = 1$, we get

$$A\,Q_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} + sa_{12} \\ a_{21} & a_{22} & a_{23} + sa_{22} \\ a_{31} & a_{32} & a_{33} + sa_{32} \end{pmatrix},$$

which is matrix A with column j added to column i times s.

To add to a column j of a matrix A another column i multiplied by s we postmultiply A by a matrix of type II in which the zero a_{ij} of the identity matrix has been multiplied by s.

Matrices of type III

A matrix of type III comes from the unit matrix exchanging two rows or columns.

$$Q_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Postmultiplying the matrix

$$A\ Q_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

by the previous matrix, whose determinant is $|Q_3| = -1$, we get

$$AQ_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{13} & a_{12} \\ a_{21} & a_{23} & a_{22} \\ a_{31} & a_{33} & a_{32} \end{pmatrix},$$

which is matrix A with the two corresponding columns exchanged.

Therefore, to exchange columns in a matrix, it is enough to post-multiply the said matrix by the corresponding matrix of type III.

Application to the orthogonalization algorithm

Consider the matrix

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-t_1}{q} & \frac{-t_2}{q} & \frac{1}{q} \end{pmatrix}.$$

If we multiply matrix A by T we get:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{-t_1}{q} & \frac{-t_2}{q} & \frac{1}{q} \end{pmatrix} = \begin{pmatrix} a_{11} - t_1\frac{a_{13}}{q} & a_{12} - t_2\frac{a_{13}}{q} & \frac{a_{13}}{q} \\ a_{21} - t_1\frac{a_{23}}{q} & a_{22} - t_2\frac{a_{23}}{q} & \frac{a_{23}}{q} \\ a_{31} - t_1\frac{a_{33}}{q} & a_{32} - t_2\frac{a_{33}}{q} & \frac{a_{33}}{q} \end{pmatrix}.$$

This matrix T remembers what the orthogonalization algorithm does in each step, since it divides the third column (pivot) by q and combines the other columns subtracting the scalar products t_1 and t_2 by the transformed pivot columns.

Therefore, each step of the orthogonalization algorithm is equivalent to multiplying by a matrix of this type.

It is interesting to note that this matrix has determinant $1/q$ and that it can be written as

$$\begin{pmatrix} a_{11} - t_1\dfrac{a_{13}}{q} & a_{12} - t_2\dfrac{a_{13}}{q} & \dfrac{a_{13}}{q} \\ a_{21} - t_1\dfrac{a_{23}}{q} & a_{22} - t_2\dfrac{a_{23}}{q} & \dfrac{a_{23}}{q} \\ a_{31} - t_1\dfrac{a_{33}}{q} & a_{32} - t_2\dfrac{a_{33}}{q} & \dfrac{a_{33}}{q} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{1}{q} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \dfrac{-t_1}{q} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{-t_2}{q} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \dfrac{-t_1}{q} & \dfrac{-t_2}{q} & \dfrac{1}{q} \end{pmatrix}$$

which shows that all the transformations of the orthogonalization algorithm can be done step by step, with the three matrices, or, at once, with the final matrix.

Determinant formula

To obtain the formula of the determinant of matrix A as a function of the pivot values, proceed as follows. Since starting from a regular matrix B instead of the unit, after pivoting with all the row vectors of the matrix and exchanging columns, the inverse matrix is obtained, it results:

$$A^{-1} = BT_1T_2 \dots T_nP_1P_2, \dots P_p,$$

where the matrices $T_1T_2 \dots T_n$ are the matrices by which you have to post-multiply, to obtain the transformed matrices, and the matrices $P_1P_2, \dots P_p$ are the matrices that exchange the columns.

As the determinant of the product is the product of the determinants it results:

$$|A^{-1}| = |B||T_1||T_2| \dots |T_n||P_1||P_2|, \dots |P_p| = |B| \prod_{i=1}^{n} \frac{1}{q_i}(-1)^p,$$

where $q_1, q_2, \dots, q_n$ are the pivots (scalar products of the pivot columns) and p is the number of exchanges.

Finally, from this equation, the following formula of the determinant results:

$$|A| = (-1)^p |B|^{-1} \prod_{i=1}^{n} q_i,$$

Matrices of type I

Premultiplying the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

by a matrix of type I, we get

$$Q_1\ A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ qa_{31} & qa_{32} & qa_{33} \end{pmatrix}$$

The result is that the row corresponding to the changed element is multiplied by q.

Therefore, to multiply a row of a given matrix by a given value, we pre-multiply it by a matrix of type I.

Matrices of type II

Premultiplying the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

by a matrix of type II, we get

$$Q_2\ A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + sa_{31} & a_{22} + sa_{32} & a_{23} + sa_{33} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

which is the initial matrix A in which one of the rows is added another row times s.

Therefore, to add to a row i of a matrix A, another row j multiplied by a value s it is enough to replace the zero of the unit matrix in the position (i, j) by the value s, that is, pre-multiply A by the corresponding type II matrix.

Matrices of type III

Premultiplying the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

by a matrix of type III, we get

$$Q_3\ A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix},$$

which is the initial matrix A in which the two rows have been exchanged .

Therefore, to exchange rows in a matrix, it is sufficient to pre-multiply that matrix by the corresponding matrix of type III.