Homogeneous linear systems of equations

Algebra

RESOLUTION OF A HOMOGENEOUS LINEAR SYSTEM

$$
\begin{array}{cccccc}
a_{11}x_1 & +a_{12}x_2 & + & \cdots & +a_{1n}x_n & = & 0 \\
a_{21}x_1 & +a_{22}x_2 & + & \cdots & +a_{2n}x_n & = & 0 \\
\cdots & \cdots & & \cdots & \cdots & \cdots & \cdots \\
a_{m1}x_1 & +a_{m2}x_2 & + & \cdots & +a_{mn}x_n & = & 0
\end{array}
$$

$$
\begin{array}{rcl}
(a_{11}, a_{12}, \cdots, a_{1n}) \cdot (x_1, x_2, \cdots, x_n) & = & 0 \\
(a_{21}, a_{22}, \cdots, a_{2n}) \cdot (x_1, x_2, \cdots, x_n) & = & 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & \cdots \\
(a_{m1}, a_{m2}, \cdots, a_{mn}) \cdot (x_1, x_2, \cdots, x_n) & = & 0
\end{array}
$$

The solution of the system is the subspace orthogonal to the subspace generated by the vectors whose components are the coefficients of the equations.

$$
\begin{array}{rrrrcl}
 & x_2 & -x_3 & +2x_4 & = & 0 \\
x_1 & +x_2 & +x_3 & & = & 0 \\
x_1 & +2x_2 & & +x_4 & = & 0 \\
x_1 & +x_2 & & +x_4 & = & 0
\end{array}
$$

	Iteration 1			
a_1	v_1	$\mathbf{v_2}$	v_3	v_4
0	1	**0**	0	0
1	0	**1**	0	0
-1	0	**0**	1	0
2	0	**0**	0	1
t	0	1	-1	2

	Iteration 2			
a_2	$\mathbf{v_1}$	w_1	v_3	v_4
1	**1**	0	0	0
1	**0**	1	1	-2
1	**0**	0	1	0
0	**0**	0	0	1
t	1	1	2	2

	Iteration 3			
a_3	w_2	w_1	v_3	$\mathbf{v_4}$
1	1	-1	-2	**2**
2	0	1	1	**-2**
0	0	0	1	**0**
1	0	0	0	**1**
t	1	1	0	1

	Iteration 4			
a_4	w_2	w_1	$\mathbf{v_3}$	w_3
1	3	1	**-2**	-2
1	-2	-1	**1**	2
0	0	0	**1**	-0
1	1	1	**0**	-1
t	2	1	1	-1

	Final table			
	w_2	w_1	w_4	w_3
	-1	-1	2	0
	0	0	1	1
	2	1	1	-1
	1	1	0	-1

$$
\begin{array}{rrrrcl}
 & x_2 & -x_3 & +2x_4 & = & 0 \\
x_1 & +x_2 & +x_3 & & = & 0 \\
x_1 & +2x_2 & & +x_4 & = & 0 \\
x_1 & +x_2 & & +x_4 & = & 0
\end{array}
$$

	Final table			
	w_2	w_1	w_4	w_3
	-1	-1	2	0
	0	0	1	1
	2	1	1	-1
	1	1	0	-1

System	Solution
$\begin{array}{rrrrcl} & x_2 & -x_3 & +2x_4 & = & 0 \\ x_1 & +x_2 & +x_3 & & = & 0 \\ x_1 & +2x_2 & & +x_4 & = & 0 \\ x_1 & +x_2 & & +x_4 & = & 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$
$\begin{array}{rrrrcl} x_1 & +x_2 & & +x_4 & = & 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \rho_1 \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + \rho_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \rho_3 \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \end{pmatrix}$
$\begin{array}{rrrrcl} x_1 & +2x_2 & & +x_4 & = & 0 \\ x_1 & +x_2 & & +x_4 & = & 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \rho_1 \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + \rho_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$
$\begin{array}{rrrrcl} & x_2 & -x_3 & +2x_4 & = & 0 \\ x_1 & +2x_2 & & +x_4 & = & 0 \\ x_1 & +x_2 & & +x_4 & = & 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \rho_1 \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$

EQUATION OF A SUBSPACE

The solution of a homogeneous system, that is, the set of vectors that satisfy the homogeneous system constitute a linear subspace. Therefore, the homogeneous system is the system of subspace equations, because only those vectors satisfy that system.

The orthogonal subspace of the subspace is the subspace generated by the vectors of the coefficients of the linear system of equations

The homogeneous system may not be in its minimum form, that is, it may have more equations than necessary. To know if this is the case, we determine if the vectors of their coefficients are linearly independent.

If, given a homogeneous system by means of its equations, a basis of it is desired, we obtain the subspace orthogonal to that of the coefficients by means of the algorithm, obtaining the desired basis.

SOME EXAMPLES OF SUBSPACES DEFINED BY THEIR EQUATIONS AND ONE OF THEIR BASES

System	Solution
$\begin{array}{rrrrcl} & x_2 & -x_3 & +2x_4 & = & 0 \\ x_1 & +x_2 & +x_3 & & = & 0 \\ x_1 & +2x_2 & & +x_4 & = & 0 \\ x_1 & +x_2 & & +x_4 & = & 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$
$\begin{array}{rrrrcl} x_1 & +x_2 & & +x_4 & = & 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \rho_1 \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + \rho_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \rho_3 \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \end{pmatrix}$
$\begin{array}{rrrrcl} x_1 & +2x_2 & & +x_4 & = & 0 \\ x_1 & +x_2 & & +x_4 & = & 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \rho_1 \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + \rho_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$
$\begin{array}{rrrrcl} & x_2 & -x_3 & +2x_4 & = & 0 \\ x_1 & +2x_2 & & +x_4 & = & 0 \\ x_1 & +x_2 & & +x_4 & = & 0 \end{array}$	$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \rho_1 \begin{pmatrix} -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$