Determining if a vector belongs to a subspace

Algebra

Notation

The subspace generated by the columns

$$\{a1, a2, \cdots, an\}$$

of the matrix A is denoted $A\rho$, where $\rho \in R$ refers to linear combinations with arbitrary real numbers or $L(a1, a2, \cdots, an)$, where L refers to linear combination.

Three ways to see if a vector belongs to a linear space

$$v \in \mathcal{L}(a_1, a_2, \cdots, a_n) \Leftrightarrow v \in \left(\mathcal{L}(a_1, a_2, \cdots, a_n)^{\perp}\right)^{\perp} \Leftrightarrow v \perp \mathcal{L}(a_1, a_2, \cdots, a_n)^{\perp}$$

1.- The first $v \in \mathcal{L}(a_1, a_2, \cdots, a_n) \equiv v \in \mathrm{A}_{\rho}$ **, the classic, considers the subspace as the set of vectors that depend linearly on the generators.**

2.- The second $v \in \left(\mathcal{L}(a_1, a_2, \cdots, a_n)^{\perp}\right)^{\perp}$ **exploits the idempotency of the orthogonality, that is, the orthogonal subspace of the orthogonal subspace is the initial one.**

3.- The third $v \perp \mathcal{L}(a_1, a_2, \cdots, a_n)^{\perp}$ **considers that a vector belongs to a subspace if and only if it is perpendicular to its orthogonal subspace.**

Example 1. To check whether the row vectors of the matrix

$$B = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & -2 \\ 1 & 1 & 1 & 1 \\ 2 & -1 & 0 & 1 \end{pmatrix}$$ **belong to the subspace** $A_\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix}$

we obtain the subspace $A_\rho^\perp = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}_\rho$

with the algorithm :

	Iteration 1			
a_1	$\mathbf{v}_1$	v_2	v_3	v_4
1	**1**	0	0	0
0	**0**	1	0	0
0	**0**	0	1	0
-1	**0**	0	0	1
t	1	0	0	-1

	Iteration 2			
a_2	w_1	v_2	$\mathbf{v}_3$	v_4
0	1	0	**0**	1
0	0	1	**0**	0
-1	0	0	**1**	0
1	0	0	**0**	1
t	0	0	-1	1

	Final table			
	w_1	v_2	w_2	v_4
	1	0	-0	1
	0	1	-0	0
	0	0	-1	1
	0	0	-0	1

If the matrix B of the vectors given by the matrix of the orthogonal generators is multiplied by the matrix of the final table, we get

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & -2 \\ 1 & 1 & 1 & 1 \\ 2 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 3 & 1 & -1 \\ -1 & 3 & 2 & 0 \end{pmatrix}.$$

If the elements of a row and the columns associated with vectors v (1 and 2) are all null, the vector of the corresponding row of B is orthogonal to the given subspace A_ρ. Otherwise, it is not. Therefore, the first two vectors belong to the subspace and the other two, do not belong.

If a vector belongs, the columns associated with vectors w (3 and 4) give the components of the vector with respect to the basis of the subspace.

	Iteration 1			
a_1	$\mathbf{v}_1$	v_2	v_3	v_4
1	**1**	0	0	0
0	**0**	1	0	0
0	**0**	0	1	0
-1	**0**	0	0	1
t	1	0	0	-1

	Iteration 2			
a_2	w_1	v_2	$\mathbf{v}_3$	v_4
0	1	0	**0**	1
0	0	1	**0**	0
-1	0	0	**1**	0
1	0	0	**0**	1
t	0	0	-1	1

	Final table			
	w_1	v_2	w_2	v_4
	1	0	-0	1
	0	1	-0	0
	0	0	-1	1
	0	0	-0	1