Observability Analysis in State Estimation: A Unified Numerical Approach

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***Abstract*—This paper provides a unified approach to observability checking, critical measurements identification, determination of observable islands, identification of irrelevant boundary injections, and pseudo-measurements selection to restore observability. A robust yet efficient algebraic technique is proposed. Several simple examples are used to illustrate the capabilities of the proposed procedure. Results from large case studies are presented to demonstrate the appropriate computational behavior of the proposed algorithms. Finally, conclusions are duly drawn.**

***Index Terms*—Linear algebra, null space, observability analysis, orthogonal transformation, power system state estimation.**

I. INTRODUCTION

A. Motivation

THE observability problem consists of identifying if a set of available measurements is enough to be able to estimate the state of an electric energy system. Observability is related to the number of measurements but also to their types and locations. If the system state is observable, it is relevant to identify these measurements that, if missing, render the state unobservable. That is, to identify each single measurement (or set of measurements) whose elimination provokes state unobservability. Those measurements are termed critical measurements. On the other hand, if the state of the system is unobservable, it is relevant to identify observable islands, i.e., those areas of the system whose respective states can be estimated; and to detect irrelevant boundary injections, i.e., those injections at buses with unobservable branches that should not be included in the estimation. Being that the state is unobservable, it is also relevant to identify which additional measurements render the system observable. That is, identify each single measurement or set of measurements (called pseudo-measurements) that added to the set of available measurements results in an observable state.

We provide a simple algebraic algorithm involving low computational burden that presents good updating characteristics. It is based on the calculus of the null space [1] using the orthogonal transformation reported in [2] and [3].

B. Literature Review and Contributions

This paper frames itself in the rich body of observability techniques for electric energy systems. Those techniques can be classified as topological [4]–[6] and algebraic [7], [8]. Note that topological techniques involve a combinatorial computational complexity, while numerical techniques do not [9]. Additional relevant references providing an algebraic approach are [10] and [11]. Reference [12] addresses the pseudo-measurement determination problem. A rather specific analysis of observability considering ampere measurement is carried out in [13] and [14] and a general analysis in [15]. For just checking observability, the recent reference [1] is relevant. Appropriate observability (and state estimation) background is provided in [16] and [17].

This paper provides a new and robust pivoting algorithm of complexity equivalent to Gauss elimination to calculate the null space of the Jacobian matrix $\mathbf{H}$. It simultaneously allows 1) checking observability, 2) identifying critical measurements, 3) determining observable islands and irrelevant boundary injections, and 4) selecting pseudo-measurements to restore observability.

The novel contributions of this paper are as follows.

1) The proposed algorithm is not based on a factorization. While other methods perform a decomposition of matrix $\mathbf{H}$ (Householder, Givens rotation, $\mathbf{QR}$, $\mathbf{LU}$, etc.), usually based on a Gauss elimination (see [15] and [18]–[20]), the orthogonal algorithm proposed in this paper does not modify matrix $\mathbf{H}$. It modifies a working matrix called $\mathbf{W}$, which initially is set to the identity matrix, throughout a pivoting process based on the dot products of the rows of matrix $\mathbf{H}$ and the columns of matrix $\mathbf{W}$. The aim of this pivoting process is to obtain the null space of the rows of the Jacobian matrix $\mathbf{H}$, which is the called orthogonal space or null space and is denoted by $\mathbf{N}$.
2) A single procedure allows determining both measurements replacements (observable case) and observable islands (nonobservable case). The working matrix $\mathbf{W}$ contains all the information required to calculate null space matrix $\mathbf{N}$, the observable islands, and the irrelevant boundary injections. While in [7] and [8], the above information is obtained in an iterative manner, matrix $\mathbf{W}$ directly provides that information.
3) Observable state variables are easily identified through the null space $\mathbf{N}$ of the Jacobian measurement matrix $\mathbf{H}$.
4) How to restore observability is also analyzed (pseudo-measurement determination). The proposed method is computationally equivalent to Gaussian elimination, being that its complexity is similar to that of the reduced row echelon form used in [11] and [20]. In addition,

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matrix $\mathbf{S}$, denoted sensitivity matrix in [15], is obtained at no extra cost.

5) A straightforward updating mechanism is provided for the case in which a measurement is lost or a measurement becomes available. In the case that one or more essential measurements are lost, no extra computations are needed to update observable islands, and irrelevant boundary injections as this information are readily available in matrix $\mathbf{W}$. If on the other hand, a new measurement is available, it is not necessary to repeat all the computations from the beginning. Starting with the previous working matrix $\mathbf{W}$, just one extra iteration of the proposed algorithm is needed to update all the results.

C. Paper Organization

This paper is organized as follows. Section II describes formally the observability problem, including observability checking, identification of critical measurements, determination of observable islands, identification of irrelevant boundary injections, and determination of pseudo-measurements to restore observability. Section III provides the proposed solution algorithm. Section IV presents several examples that illustrate how the proposed algorithm works. Section V gives results from a case study based on the IEEE Reliability Test System [21] and from a larger network based on the IEEE 300-bus system [22]. In Section VI, some relevant conclusions are given. An Appendix provides further details of the proposed algorithm.

II. FORMULATION

Consider the measurement vector equation

$$\mathbf{h}(\mathbf{x}) = \mathbf{z} \tag{1}$$

where

$\mathbf{h}(\cdot)$ m-dimensional measurement nonlinear function;
$\mathbf{x}$ n-dimensional state vector;
$\mathbf{z}$ m-dimensional constant vector of actual measurements.

The state vector includes voltage magnitudes for all buses and voltage angles for all buses but the reference one. The measurement function may include voltage magnitudes, active and reactive power injections, active and reactive power flows, and eventually line currents. Usually $m \gg n$, but some sub-areas of the system may lack sufficient measurements to guarantee observability, while others may include more measurements than those needed to achieve observability. For observability purposes and without loss of generality [17], (1) can be linearized; thus

$$\mathbf{H}\mathbf{x} = \mathbf{z} \tag{2}$$

where $\mathbf{H}$ is the $m \times n$ Jacobian measurement matrix. In the same vein, we denominate $\overline{\mathbf{H}}$ the Jacobian measurement matrix of all plausible measurements not included in $\mathbf{H}$. The problems dealt with in this paper are as follows.

P 1) Determine if a subset of the available measurements represented by $\mathbf{H}$ is sufficient to determine the values of all state variables, i.e., if the state variables are uniquely observable given $\mathbf{H}$.

P 2) If the set of available measurements represented by $\mathbf{H}$ makes the problem observable, determine which critical measurements cannot be removed to keep the problem observable.

P 3) Assuming that the state is observable given $\mathbf{H}$, determine the set of redundant measurements in $\overline{\mathbf{H}}$ that can be used to replace a given measurement in $\mathbf{H}$ so that the problem remains observable.

P 4) If the state is not observable given $\mathbf{H}$, identify observable islands and observable state variables.

P 5) If the state is not observable, identify irrelevant boundary injections.

P 6) If all state variables are not observable given $\mathbf{H}$, obtain the minimum subset of pseudo-measurements in $\overline{\mathbf{H}}$ such that they make all state variables observable.

P 7) Specify the minimal data required to update the results if initial measurements are lost or new measurements become available.

III. ALGORITHM DESCRIPTION

In this section, we describe an algorithm to address the seven problems above. The basis for this algorithm is contained in the orthogonal transformation algorithm provided in [2] and [3] but adequately modified to solve the observability problems above.

A. Rationale

The rationale for the algorithm below is basically to express the observable state variables as a function of the available measurements. That is, to transfer "columns to rows" in $\mathbf{H}$. If all state variables can be expressed as linear combinations of measurements, the state is observable; otherwise, it is not. The actual operations are based on the orthogonal transformation algorithm reported in [2] and [3]. The proposed algorithm provides two vectors and two matrices of interest.

1) Integer vector $\mathbf{U}$ of dimension $n \times 1$ contains the list of required measurements to attain state observability (observability of all variables). These measurements are denoted *essential measurements*.
2) Integer vector $\mathbf{V}$ of dimension $(m-n)\times 1$ contains the list of *redundant measurements* for observability purposes, that is, measurements that if lost do not render the state unobservable.
3) Matrix $\mathbf{S}$ of dimension $(m-n) \times n$ contains the linear combinations of the redundant measurements in terms of the required (essential) ones. This matrix is denoted *sensitivity matrix* in [15].
4) Matrix $\mathbf{W}$ of dimension $n \times n$ contains relevant information on observable islands and irrelevant boundary injections if the state is not observable or if the state becomes unobservable as a result of the loss of measurements.

B. Algorithm

The algorithm works as follows.

Step 1) Set matrix $\mathbf{W} = \mathbf{I}_n$ (the identity matrix of dimension n), vector $\mathbf{U} = \mathbf{0}_n$ (null vector of dimension

n), and initialize vector $\mathbf{V}$ and matrix $\mathbf{S}$ to the empty sets. Let $i = 1$.

Step 2) Calculate the dot products $t_j^i = \mathbf{h}_i^T \mathbf{w}_j; j = 1, \ldots, n$, that is, the dot products of the row i of matrix $\mathbf{H}$ by the columns of $\mathbf{W}$. Note that if $\mathbf{H}$ is sparse, sparsity techniques can be used, and $\mathbf{W}$ remains sparse.

Step 3) Locate the first nonnull t_p^i corresponding to a null element in position p on vector $\mathbf{U}$, which indicate the column p of matrix $\mathbf{W}$ to pivot, and replace the pth component of vector $\mathbf{U}$ by i. If there is no such column, append i to vector $\mathbf{V}$ and vector $\mathbf{t}^T$ to matrix $\mathbf{S}$ and go to Step 6). Otherwise, continue with Step 4).

Step 4) Divide the p column of matrix $\mathbf{W}$ by t_p^i.

Step 5) For $j = 1$ to n, $j \neq p$, and $t_j^i \neq 0$, do $w_{kj} = w_{kj} - t_j^i w_{kp}$ for $k = 1, \ldots, n$.

Step 6) If $i = m$, continue with Step 7). Otherwise, increase i in one unit and continue with Step 2).

Step 7) If $\mathbf{U}$ contains only one zero element, the state is observable; otherwise, it is not. Return vectors $\mathbf{U}$ and $\mathbf{V}$ and matrices $\mathbf{S}$ and $\mathbf{W}$.

A matrix formulation of the proposed algorithm is provided in the Appendix.

C. *On Computational Complexity and Sparsity*

To improve numerical stability, while processing an active power measurement (row), a voltage angle (column) pivot should be selected, while if processing a reactive power or voltage magnitude measurement (row), a voltage magnitude (column) pivot should be used. The complexity of the proposed algorithm is proportional to 2 mn^2, which is reasonable, even to analyze large-scale systems. Vectors $\mathbf{U}$ and $\mathbf{V}$ are integer index vectors whose storage and handling is straightforward. Working matrix $\mathbf{W}$ keeps the sparsity of the Jacobian matrix $\mathbf{H}$. The pivoting process on matrix $\mathbf{W}$ presented in this paper is based on the Jacobian matrix $\mathbf{H}$, which is sparser than the square gain matrix. The proposed algorithm preserves this sparsity, as it is based on the dot products of the rows of matrix $\mathbf{H}$ and the columns of matrix $\mathbf{W}$, initially set to the identity matrix. The proposed algorithm modifies the columns of $\mathbf{W}$ to obtain the null space if the corresponding dot product is nonnull. If $\mathbf{H}$ is sparse, most dot products are null, and the corresponding columns are not modified; hence, sparsity is preserved.

D. *Solution*

From the information provided by the algorithm above, the solution of each of the seven problems formulated in Section II is stated below.

P 1) The matrix $\mathbf{H}$ corresponding with the available measurements is used as input in the algorithm, and then the vector $\mathbf{U}$ is tested for zero elements; if more than one zero element exists, the state is not observable given $\mathbf{H}$; otherwise, it is.

P 2) The set $\mathbf{H}$ of the available measurements is used as input in the algorithm, and then the indexes of the null columns of matrix $\mathbf{S}$ give the critical measurements contained in $\mathbf{U}$ that cannot be replaced by other measurements.

TABLE I
ILLUSTRATION OF THE UPDATING PROCEDURE

Initial matrix S					
	$\mathbf{h}_1$	$\ldots$	$\mathbf{h}_j$	$\ldots$	$\mathbf{h}_n$
$\mathbf{h}_1^*$	s_{11}	$\cdots$	s_{1j}	$\cdots$	s_{1n}
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\ldots$	$\vdots$
$\mathbf{h}_i^*$	s_{i1}	$\cdots$	s_{ij}	$\cdots$	s_{in}
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\ldots$	$\vdots$
$\mathbf{h}_r^*$	s_{r1}	$\cdots$	s_{rj}	$\cdots$	s_{rn}
Updated matrix S					
	$\mathbf{h}_1$	$\ldots$	$\mathbf{h}_i$	$\ldots$	$\mathbf{h}_n$
$\mathbf{h}_1^*$	$s_{11} - \frac{s_{i1}}{s_{ij}} s_{1j}$	$\cdots$	$\frac{s_{1j}}{s_{ij}}$	$\cdots$	$s_{1n} - \frac{s_{in}}{s_{ij}} s_{1j}$
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\ldots$	$\vdots$
$\mathbf{h}_j^*$	$-\frac{s_{i1}}{s_{ij}}$	$\cdots$	$\frac{1}{s_{ij}}$	$\cdots$	$-\frac{s_{in}}{s_{ij}}$
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\ldots$	$\vdots$
$\mathbf{h}_r^*$	$s_{r1} - \frac{s_{i1}}{s_{ij}} s_{rj}$	$\cdots$	$\frac{s_{rj}}{s_{ij}}$	$\cdots$	$s_{rn} - \frac{s_{in}}{s_{ij}} s_{rj}$

P 3) The nonnull elements s_{ij} of matrix $\mathbf{S}$ give the replacements that preserve observability (see Table I). If $s_{ij} \neq 0$, the measurement $\mathbf{h}_i^*$ can replace the measurement $\mathbf{h}_j$. In this table, $\mathbf{h}$ and $\mathbf{h}^*$ refer to the essential and redundant measurements, respectively, and $r = m - n$. In the upper part of Table I, the first column corresponds to the $\mathbf{h}^*$ redundant measurements and the first row corresponds to the $\mathbf{h}$ essential measurements. The elements s_{ij} of matrix $\mathbf{S}$ are the linear combination coefficients of the redundant measurements in terms of the essential ones

$$\mathbf{h}_i^* = s_{i1}\mathbf{h}_1 + \ldots + s_{ij}\mathbf{h}_j + \ldots + s_{in}\mathbf{h}_n.$$

The nonnull elements s_{ij} of matrix $\mathbf{S}$ provide the replacements that preserve observability. If $s_{ij} \neq 0$, measurement $\mathbf{h}_i^*$ can replace measurement $\mathbf{h}_j$

$$\mathbf{h}_j = -\frac{s_{i1}}{s_{ij}}\mathbf{h}_1 - \ldots + \frac{1}{s_{ij}}\mathbf{h}_i^* - \ldots - \frac{s_{in}}{s_{ij}}\mathbf{h}_n.$$

Performing the above transformation, the jth essential measurement becomes redundant, and the ith redundant measurement becomes essential. The rest of transformations required for updating matrix $\mathbf{S}$ are shown in the lower part of Table I.

P 4) Defining matrix $\mathbf{N}$ (called test matrix in [10]) as the matrix containing the columns of $\mathbf{W}$ corresponding with null components in $\mathbf{U}$, variables in the same observable island have identical rows in $\mathbf{N}$. However, variables within different observable islands might have identical rows in $\mathbf{N}$. The columns of matrix $\mathbf{N}$ generate the null space of matrix $\mathbf{H}$ (see [2]). Defining matrix $\mathbf{C}$ as $\mathbf{C} = \mathbf{AN}$, where $\mathbf{A}$ is the branch node incidence matrix, nonnull rows in this matrix $\mathbf{C}$ indicate unobservable branches. Removing unobservable branches, observable islands are identified. It should be noted that observable islands provide insightful information to identify new pseudo-measurements

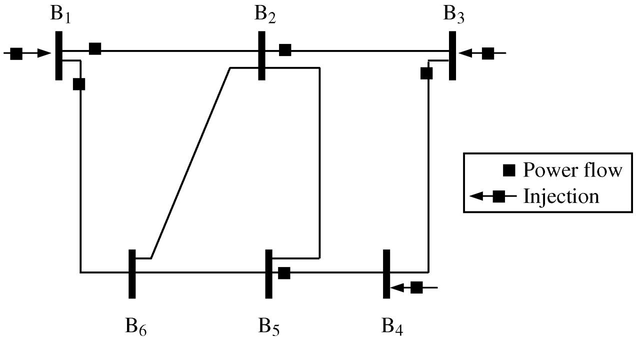

Fig. 1. Measurement configuration of a six-bus system.

to restore observability. The null rows of matrix $\mathbf{N}$ correspond to the observable state variables.

P 5) The irrelevant boundary injection measurements are injections at buses with unobservable branches. If we remove them from the available set of measurements, the set of observable islands remains unaltered. These injections should not be included in the estimation process as they involve unobservable variables. In the proposed algorithm, the irrelevant boundary injection measurements can be detected as follows: considering matrix $\mathbf{N}$ expanded with the columns of $\mathbf{W}$ corresponding with the indexes in $\mathbf{U}$ associated with irrelevant boundary injections, the new matrix has the same equal rows than matrix $\mathbf{N}$, and as a consequence, they define the same set of observable islands.

P 6) Measurements corresponding to $\left[\frac{\mathbf{H}}{\overline{\mathbf{H}}}\right]$ are used as input to the algorithm, and then the vector $\mathbf{U}$ is tested for zero elements; if more than one exist, the state is not observable given $\left[\frac{\mathbf{H}}{\overline{\mathbf{H}}}\right]$; otherwise, it is, and the set of pseudo-measurements additionally required for observability are those corresponding to the rows of $\overline{\mathbf{H}}$ whose indexes are contained in $\mathbf{U}$.

P 7) The minimal information to update results consists of the output of the proposed algorithm, that is, vectors $\mathbf{U}$ and $\mathbf{V}$ and matrices $\mathbf{S}$ and $\mathbf{W}$. Particularly, if one or more measurements are lost, and the system becomes unobservable, matrix $\mathbf{N}$ can be updated adding the columns of the working matrix $\mathbf{W}$ corresponding to the indexes in $\mathbf{U}$ associated to the lost measurements, and then, it is straightforward to calculate the new observable islands and the irrelevant measurement injections from the new matrix $\mathbf{N}$ following the procedure in item P 4) above. If a new measurement is added, it is not necessary to repeat all the computations from the beginning. Starting with the previous working matrix $\mathbf{W}$, just one extra iteration of the proposed algorithm is needed to update all the results.

IV. EXAMPLES

The example considered in [17, p. 92] is used in this section to illustrate the proposed technique. Fig. 1 depicts the considered system and the measurement configuration. The measurement set contains active power flows and active power injections. For the measurement configuration in Fig. 1, the Jacobian measurement matrix $\mathbf{H}$ is

$$\begin{array}{c} \\ P_1 \\ P_3 \\ P_4 \\ P_{3,4} \\ P_{1,2} \\ P_{1,6} \\ P_{5,4} \\ P_{2,3} \end{array}
\begin{array}{c} \begin{array}{cccccc} \theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 & \theta_6 \end{array} \\
\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

where P_k represents the active power injection at bus k and $P_{i,j}$ the power flow through line $i-j$. Columns 1–6 corresponds to voltage angles 1–6 (state variables). Several cases are analyzed below.

A. Observable Case With No Critical Measurements

If the Jacobian measurement matrix is

$$\mathbf{H} = \begin{array}{c} P_1 \\ P_3 \\ P_4 \\ P_{3,4} \\ P_{1,2} \\ P_{1,6} \\ P_{5,4} \\ P_{2,3} \end{array}
\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

the algorithm works as stated in Table II. For the ith iteration, vector $\mathbf{U}$ corresponds with the first row of the table, dot products (denoted as $\mathbf{t}^i$) are shown in last row, the ith row of matrix $\mathbf{H}$ (denoted as $\mathbf{h}_i$) is shown as the first column of the table, the working matrix $\mathbf{W}$ constitutes the central part of the table, and the pivot column is boldfaced.

Note that in the first iteration, measurement P_1 has been introduced to the pivoting process in the first column of the table. Initially, vector $\mathbf{U}^T$ is [0 0 0 0 0 0] and the initial working matrix $\mathbf{W}$ is the identity matrix. In the table of iteration 2, $\mathbf{U}^T$ has been updated to [1 0 0 0 0 0] because the first column of matrix $\mathbf{W}$ has been used as pivot, matrix $\mathbf{W}$ has been modified according to the algorithm proposed, and measurement P_3 has been introduced in the first column of the table to perform the next iteration of the pivoting process. The previous steps are repeated for all the available measurements corresponding with matrix $\mathbf{H}$. Note that measurements $P_{1,6}$, $P_{5,4}$, and $P_{2,3}$ in iterations 6–8 are redundant because dot products corresponding with zero entries in vector $\mathbf{U}$ are nulls. In the Final Table, vector $\mathbf{U}^T$ is [1 4 2 3 5 | 0]. The vector $\mathbf{V}^T$ is [6 7 8] and contains the indexes of the redundant measurements $P_{1,6}$, $P_{5,4}$, and $P_{2,3}$. The sensitivity matrix $\mathbf{S}$ is obtained taking the dot product vectors $\mathbf{t}_6$, $\mathbf{t}_7$, and $\mathbf{t}_8$ associated with them

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & | & 0 \\ 0 & -1 & 0 & -1 & 0 & | & 0 \\ 0 & 1 & -1 & 0 & 0 & | & 0 \end{bmatrix}.$$

TABLE II
ITERATIONS OF ALGORITHM III-B IN EXAMPLE IV-A

Iteration 1						
$\mathbf{h}_1$	0	0	0	0	0	0
2	**1**	0	0	0	0	0
-1	**0**	1	0	0	0	0
0	**0**	0	1	0	0	0
0	**0**	0	0	1	0	0
0	**0**	0	0	0	1	0
-1	**0**	0	0	0	0	1
$\mathbf{t}^1$	**2**	-1	0	0	0	-1

Iteration 2						
$\mathbf{h}_2$	1	0	0	0	0	0
0	1/2	1/2	**0**	0	0	1/2
-1	0	1	**0**	0	0	0
2	0	0	**1**	0	0	0
-1	0	0	**0**	1	0	0
0	0	0	**0**	0	1	0
0	0	0	**0**	0	0	1
$\mathbf{t}^2$	0	-1	**2**	-1	0	0

Iteration 3						
$\mathbf{h}_3$	1	0	2	0	0	0
0	1/2	1/2	0	**0**	0	1/2
0	0	1	0	**0**	0	0
-1	0	1/2	1/2	**1/2**	0	0
2	0	0	0	**1**	0	0
-1	0	0	0	**0**	1	0
0	0	0	0	**0**	0	1
$\mathbf{t}^3$	0	-1/2	-1/2	**3/2**	-1	0

Iteration 4						
$\mathbf{h}_4$	1	0	2	3	0	0
0	1/2	**1/2**	0	0	0	1/2
0	0	**1**	0	0	0	0
1	0	**2/3**	2/3	1/3	1/3	0
-1	0	**1/3**	1/3	2/3	2/3	0
0	0	**0**	0	0	1	0
0	0	**0**	0	0	0	1
$\mathbf{t}^4$	0	**1/3**	1/3	-1/3	-1/3	0

Iteration 5						
$\mathbf{h}_5$	1	4	2	3	0	0
1	1/2	3/2	-1/2	1/2	**1/2**	1/2
-1	0	3	-1	1	**1**	0
0	0	2	0	1	**1**	0
0	0	1	0	1	**1**	0
0	0	0	0	0	**1**	0
0	0	0	0	0	**0**	1
$\mathbf{t}^5$	1/2	-3/2	1/2	-1/2	**-1/2**	1/2

Iteration 6						
$\mathbf{h}_6$	1	4	2	3	5	0
1	1	0	0	0	-1	1
0	1	0	0	0	-2	1
0	1	-1	1	0	-2	1
0	1	-2	1	0	-2	1
0	1	-3	1	-1	-2	1
-1	0	0	0	0	0	1
$\mathbf{t}^6$	1	0	0	0	-1	0

Iteration 7						
$\mathbf{h}_7$	1	4	2	3	5	0
0	1	0	0	0	-1	1
0	1	0	0	0	-2	1
0	1	-1	1	0	-2	1
-1	1	-2	1	0	-2	1
1	1	-3	1	-1	-2	1
0	0	0	0	0	0	1
$\mathbf{t}^7$	0	-1	0	-1	0	0

Iteration 8						
$\mathbf{h}_8$	1	4	2	3	5	0
0	1	0	0	0	-1	1
1	1	0	0	0	-2	1
-1	1	-1	1	0	-2	1
0	1	-2	1	0	-2	1
0	1	-3	1	-1	-2	1
0	0	0	0	0	0	1
$\mathbf{t}^8$	0	1	-1	0	0	0

Final Table					
1	4	2	3	5	0
1	0	0	0	-1	1
1	0	0	0	-2	1
1	-1	1	0	-2	1
1	-2	1	0	-2	1
1	-3	1	-1	-2	1
0	0	0	0	0	1

As vectors $\mathbf{U}$ and $\mathbf{V}$ identify measurements, they can also be expressed as

$$\begin{aligned}\mathbf{U}^T &= [\,P_1 \quad P_{3,4} \quad P_3 \quad P_4 \quad P_{1,2} \quad | \quad 0\,]\\ \mathbf{V}^T &= [P_{1,6} \quad P_{5,4} \quad P_{2,3}].\end{aligned}$$

The above information can be written in a convenient tableau format as follows (note that the last component of vector $\mathbf{U}$ and the last column of matrix $\mathbf{S}$ have been removed)

$$\begin{array}{c|ccccc} & P_1 & P_{3,4} & P_3 & P_4 & P_{1,2} \\ \hline P_{1,6} & 1 & 0 & 0 & 0 & -1 \\ P_{5,4} & 0 & -1 & 0 & -1 & 0 \\ P_{2,3} & 0 & 1 & -1 & 0 & 0 \end{array}$$

where the column indexes of matrix $\mathbf{S}$ (on top of it) are the elements of vector $\mathbf{U}$, whereas row indexes (left-hand side) are the elements of vector $\mathbf{V}$. Matrix $\mathbf{S}$ contains the linear combinations of the redundant measurements in terms of the required (essential) ones. As vector $\mathbf{U}$ has only one zero element, the state is observable. Moreover, as matrix $\mathbf{S}$ contains only one zero column (the one corresponding with the zero element in $\mathbf{U}$), no critical measurement exists. Note that any measurement contained in $\mathbf{U}$ can be replaced by a measurement contained in $\mathbf{V}$, provided that the corresponding element in matrix $\mathbf{S}$ is not null. For example, measurement $P_{3,4}$ can be replaced by either measurement $P_{5,4}$ or measurement $P_{2,3}$ but not by measurement $P_{1,6}$. If measurement $P_{3,4}$ is lost and it is replaced by measurement $P_{5,4}$, the updated tableau becomes (see Table I)

$$\begin{array}{c|ccccc} & P_1 & P_{5,4} & P_3 & P_4 & P_{1,2} \\ \hline P_{1,6} & 1 & 0 & 0 & 0 & -1 \\ P_{2,3} & 0 & -1 & -1 & -1 & 0 \end{array}.$$

B. Observable Case With Critical Measurements

If the Jacobian measurement matrix is (measurement $P_{2,3}$ lost)

$$\mathbf{H} = \begin{array}{c} P_1 \\ P_3 \\ P_4 \\ P_{3,4} \\ P_{1,2} \\ P_{1,6} \\ P_{5,4} \end{array} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

the tableau provided by the proposed algorithm is

$$\begin{array}{c|ccccc} & P_1 & P_{3,4} & P_3 & P_4 & P_{1,2} \\ \hline P_{1,6} & 1 & 0 & 0 & 0 & -1 \\ P_{5,4} & 0 & -1 & 0 & -1 & 0 \end{array}.$$

As vector

$$\mathbf{U}^T = [\,P_1 \quad P_{3,4} \quad P_3 \quad P_4 \quad P_{1,2} \quad | \quad 0\,]$$

has only one zero element, the state is observable. However, as column 3 in matrix $\mathbf{S}$ is a null column, measurement P_3 is a critical measurement.

C. Unobservable Case, Observable Islands and Pseudo-Measurements Determination to Restore Observability

Consider a measurement Jacobian matrix $\mathbf{H}$ containing measurements P_4, $P_{3,4}$, and $P_{1,2}$ and the complementary matrix $\overline{\overline{\mathbf{H}}}$ containing the rest of measurements

$$\mathbf{H} = \begin{array}{c} P_4 \\ P_{3,4} \\ P_{1,2} \end{array} \begin{bmatrix} 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overline{\overline{\mathbf{H}}} = \begin{array}{c} P_1 \\ P_3 \\ P_{1,6} \\ P_{5,4} \\ P_{2,3} \end{array} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

Applying the proposed algorithm to matrix $\mathbf{H}$, vector $\mathbf{U}$ becomes $[P_{1,2}\ 0\ P_{3,4}\ P_4\ 0\ 0]$; therefore, the state is not observable as $\mathbf{U}$ contains more than one null element. Matrix

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

TABLE III
ITERATIONS OF ALGORITHM III-B IN EXAMPLE IV-C

Iteration 1

$\mathbf{h}_1$	0	0	0	0	0	0
0	1	0	0	**0**	0	0
0	0	1	0	**0**	0	0
-1	0	0	1	**0**	0	0
2	0	0	0	**1**	0	0
-1	0	0	0	**0**	1	0
0	0	0	0	**0**	0	1
$\mathbf{t}^1$	0	0	-1	**2**	-1	0

Iteration 2

$\mathbf{h}_2$	0	0	0	1	0	0
0	1	0	**0**	0	0	0
0	0	1	**0**	0	0	0
1	0	0	**1**	0	0	0
-1	0	0	**1/2**	1/2	1/2	0
0	0	0	**0**	0	1	0
0	0	0	**0**	0	0	1
$\mathbf{t}^2$	0	0	**1/2**	-1/2	-1/2	0

Iteration 3

$\mathbf{h}_3$	0	0	2	1	0	0
1	**1**	0	0	0	0	0
-1	**0**	1	0	0	0	0
0	**0**	0	2	1	1	0
0	**0**	0	1	1	1	0
0	**0**	0	0	0	1	0
0	**0**	0	0	0	0	1
$\mathbf{t}^3$	**1**	-1	0	0	0	0

Final Table

3	0	2	1	0	0
1	1	0	0	0	0
0	1	0	0	0	0
0	0	2	1	1	0
0	0	1	1	1	0
0	0	0	0	1	0
0	0	0	0	0	1

is directly provided by the algorithm (see Table III, Final Table). Selecting the columns in $\mathbf{W}$ corresponding with null elements in $\mathbf{U}$, matrix $\mathbf{N}$ below is obtained

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that the rows of matrix $\mathbf{N}$ correspond to the state variables. Since there are no null rows, then there are not observable variables. However, variables with the same row values are simultaneously observable. Matrix $\mathbf{C}$ is then calculated as

$$\mathbf{C} = \begin{array}{r} \text{Branch}: 1-2 \\ \text{Branch}: 1-6 \\ \text{Branch}: 2-3 \\ \text{Branch}: 2-5 \\ \text{Branch}: 2-6 \\ \text{Branch}: 3-4 \\ \text{Branch}: 4-5 \\ \text{Branch}: 5-6 \end{array} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

Null rows in matrix $\mathbf{C}$ indicate that branches 1–2, 3–4, and 4–5 are observable, while the remaining ones are not. Removing the unobservable branches, the observable islands {1,2}, {3,4,5}, and {6} are obtained, as it is shown in Fig. 2. Moreover, if measurements $P_{1,2}$ and $P_{3,4}$ are lost, the matrix $\mathbf{N}$ can be updated without extra iterations. Adding to the previous $\mathbf{N}$, the columns of $\mathbf{W}$ (see Table III, Final Table) corresponding to the indexes 2 and 3 of $\mathbf{U}$ associated with the lost measurements, the updated $\mathbf{N}$ matrix becomes

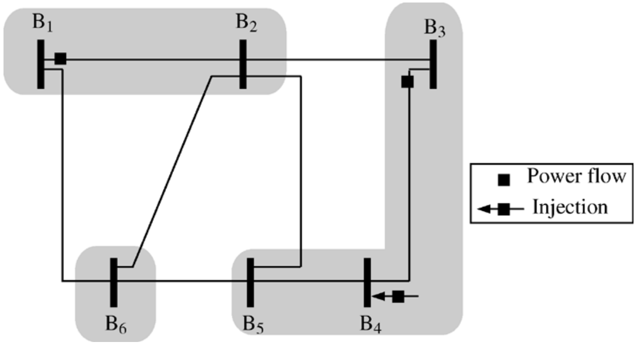

Fig. 2. Observable islands in the six-bus system.

$$\mathbf{N} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which indicates that all buses are isolated islands because all rows in matrix $\mathbf{N}$ are different. Pseudo-measurements to restore observability are identified below. In order to restore observability, at least two extra measurements are required (measurements $P_{1,2}$ and $P_{3,4}$ are not lost). Therefore, measurements corresponding to matrix $\overline{\mathbf{H}}$ are considered.

If the proposed algorithm is applied to the Jacobian measurement matrix $\left[\frac{\mathbf{H}}{\overline{\mathbf{H}}}\right]$, the following tableau is obtained:

$$\begin{array}{c|ccccc} & P_{1,2} & P_4 & P_{3,4} & P_3 & P_1 \\ \hline P_{1,6} & -1 & 0 & 0 & 0 & 1 \\ P_{5,4} & 0 & -1 & -1 & 0 & 0 \\ P_{2,3} & 0 & 0 & 1 & -1 & 0 \end{array}.$$

Note that this tableau is identical to the initial one but with the columns reordered. Measurements in $\mathbf{U}$ belonging to $\overline{\mathbf{H}}$ are the pseudo-measurements needed to restore observability (in this particular case, P_1 and P_3). However, since essential measurements can be substituted by redundant ones, some of the redundant measurements ($P_{1,6}$, $P_{5,4}$, and $P_{2,3}$) can be used to restore observability instead of essential measurements P_1 and P_3.

It should be noted that the treatment of irrelevant boundary injections are considered in the case study below.

V. CASE STUDY

A. IEEE RTS 24-Bus System

The IEEE Reliability Test System [21], depicted in Fig. 3, is used to illustrate the proposed methodology. Consider that the following measurements are provided: active power injections at buses 2, 7, 10, 13, 15, 19, 22, and 24 and active power flows at the sending end of lines 3-24, 4-2, 5-1, 10-8, 12-9, 12-10, 16-14, 20-19, and 21-22. The Jacobian measurement matrix $\mathbf{H}$ has 24

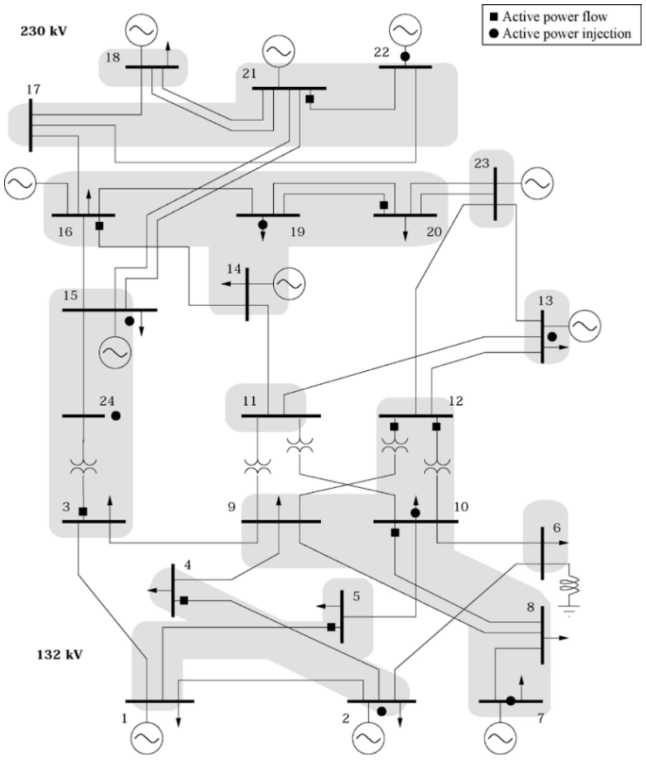

Fig. 3. Configuration of the IEEE RTS 24-bus system, initial measurement set, and observable islands.

columns corresponding with 24 phases angles and 17 rows corresponding with the measurements available. The proposed algorithm identifies the state as unobservable, and no redundant measurement is found. Matrix $\mathbf{N}$ becomes

$$\mathbf{N}=\begin{bmatrix}
1.00 & 0 & 0 & 0 & 0 & 0 & 0\\
0.93 & 0.06 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0.26 & 0 & 0.74 & 0\\
0.93 & 0.06 & 0 & 0 & 0 & 0 & 0\\
1.00 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1.00 & 0 & 0 & 0 & 0 & 0\\
0.28 & 0.41 & 0.31 & 0 & 0 & 0 & 0\\
0.28 & 0.41 & 0.31 & 0 & 0 & 0 & 0\\
0.28 & 0.41 & 0.31 & 0 & 0 & 0 & 0\\
0.28 & 0.41 & 0.31 & 0 & 0 & 0 & 0\\
0 & 0 & 1.00 & 0 & 0 & 0 & 0\\
0.28 & 0.41 & 0.31 & 0 & 0 & 0 & 0\\
0.11 & 0.16 & 0.51 & 0 & 0 & 0 & 0.22\\
0 & 0 & 0 & 0 & 0 & 1.00 & 0\\
0 & 0 & 0 & 0.26 & 0 & 0.74 & 0\\
0 & 0 & 0 & 0 & 0 & 1.00 & 0\\
0 & 0 & 0 & 1.00 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1.00 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1.00 & 0\\
0 & 0 & 0 & 0 & 0 & 1.00 & 0\\
0 & 0 & 0 & 1.00 & 0 & 0 & 0\\
0 & 0 & 0 & 1.00 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1.00\\
0 & 0 & 0 & 0.26 & 0 & 0.74 & 0
\end{bmatrix}.$$

Matrix $\mathbf{C}$ is then calculated as

$$\mathbf{C}=\begin{array}{l}
\mathbf{Branch:1-2}\\
\mathbf{Branch:1-3}\\
\mathrm{Branch:1-5}\\
\mathrm{Branch:2-4}\\
\mathbf{Branch:2-6}\\
\mathbf{Branch:9-3}\\
\mathrm{Branch:24-3}\\
\mathbf{Branch:9-4}\\
\mathbf{Branch:10-5}\\
\mathbf{Branch:10-6}\\
\mathrm{Branch:7-8}\\
\mathrm{Branch:9-8}\\
\mathrm{Branch:10-8}\\
\mathbf{Branch:11-9}\\
\mathrm{Branch:12-9}\\
\mathbf{Branch:11-10}\\
\mathbf{Branch:12-10}\\
\mathbf{Branch:13-11}\\
\mathbf{Branch:14-11}\\
\mathbf{Branch:13-12}\\
\mathbf{Branch:23-12}\\
\mathbf{Branch:13-23}\\
\mathrm{Branch:16-14}\\
\mathbf{Branch:16-15}\\
\mathbf{Branch:21-15}\\
\mathrm{Branch:15-24}\\
\mathbf{Branch:17-16}\\
\mathrm{Branch:19-16}\\
\mathbf{Branch:18-17}\\
\mathrm{Branch:22-17}\\
\mathbf{Branch:18-21}\\
\mathrm{Branch:19-20}\\
\mathbf{Branch:23-20}\\
\mathrm{Branch:21-22}
\end{array}
\begin{bmatrix}
0.06 & -0.06 & 0 & 0 & 0 & 0 & 0\\
1.00 & 0 & 0 & -0.26 & 0 & -0.74 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0.93 & -0.93 & 0 & 0 & 0 & 0 & 0\\
0.28 & 0.41 & 0.31 & -0.26 & 0 & -0.74 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
-0.66 & 0.34 & 0.31 & 0 & 0 & 0 & 0\\
-0.72 & 0.41 & 0.31 & 0 & 0 & 0 & 0\\
0.28 & -0.59 & 0.31 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
-0.28 & -0.41 & 0.69 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
-0.28 & -0.41 & 0.69 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0.11 & 0.16 & -0.49 & 0 & 0 & 0 & 0.22\\
0 & 0 & -1.00 & 0 & 0 & 1.00 & 0\\
-0.17 & -0.25 & 0.20 & 0 & 0 & 0 & 0.22\\
-0.28 & -0.41 & -0.31 & 0 & 0 & 0 & 1.00\\
0.12 & 0.16 & 0.51 & 0 & 0 & 0 & -0.78\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -0.26 & 0 & 0.26 & 0\\
0 & 0 & 0 & 0.74 & 0 & -0.74 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1.00 & 0 & -1.00 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1.00 & 1.00 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1.00 & 1.00 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1.00 & 1.00\\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Removing the unobservable branches identified by the nonnull rows of matrix $\mathbf{C}$ (indicated in boldface), the observable islands are obtained. These islands are {1,5}, {2,4}, {3,15,24}, {6}, {7,8,9,10,12}, {11}, {13}, {14,16,19,20}, {17,21,22}, {18} and {23} (see Fig. 3).

From matrix $\mathbf{W}$, all the irrelevant boundary injections are identified in one step. The irrelevant boundary injections are injections at buses with unobservable branches. In this example, the irrelevant injections are P_2, P_{10}, P_{13}, and P_{15}

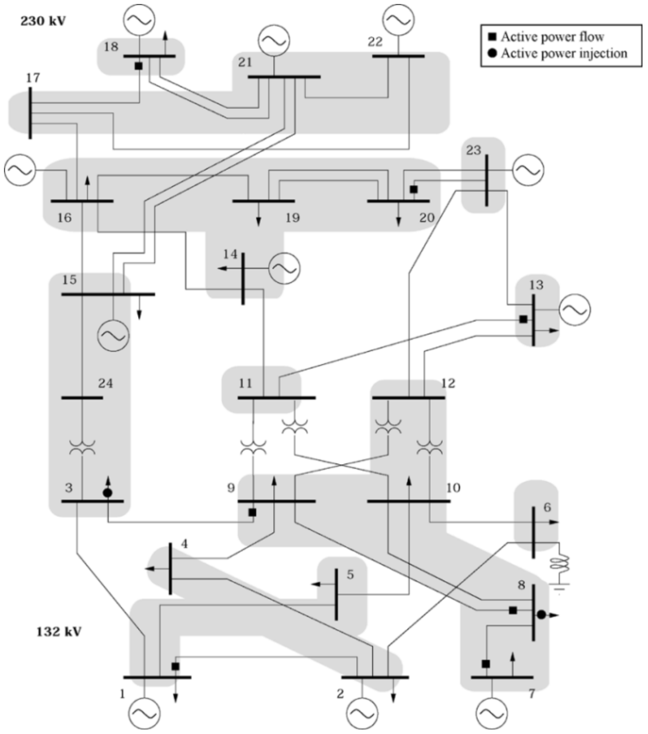

Fig. 4. IEEE RTS 24-bus system. Additional measurements (essential and redundant) to restore observability.

because taking the matrix $\mathbf{W}^1$ composed by the columns of $\mathbf{W}$ associated with them

$$\mathbf{W}^1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.01 \\ 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.03 & 0 & 0 \\ 0 & 0.03 & 0 & 0 \\ 0 & 0.03 & 0 & 0 \\ 0 & 0.03 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.03 & 0 & 0 \\ 0 & 0.01 & 0.02 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 001 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.01 \end{bmatrix}$$

matrix $[\mathbf{N}|\mathbf{W}^1]$ has the same sets of equal rows than matrix $\mathbf{N}$, and consequently, they define the same observable islands.

To have a full-rank Jacobian measurement matrix, i.e., to attain observability, the minimum set of new measurements required is 6. For example, this is achieved including the active power injection at bus 3 and active power flows at the sending ends of lines 1-2, 13-11, 9-3, 18-17, and 20-23. This is illustrated in Fig. 4. Additionally, consider that redundant measurements related with bus 8 are added: active power injection at bus 8 and active power flows at the sending end of lines 7-8 and 8-9. All the above measurements are indicated in Fig. 4. The proposed algorithm provides the following information:

$$\mathbf{U} = \begin{bmatrix} P_{5,1} \\ P_2 \\ P_{3,24} \\ P_{4,2} \\ P_{9,3} \\ P_{1,2} \\ P_7 \\ P_{10,8} \\ P_{12,9} \\ P_{10} \\ P_{13,11} \\ P_{12,10} \\ P_{13} \\ P_{16,14} \\ P_{15} \\ P_{20,19} \\ P_{18,17} \\ P_{20,23} \\ P_{19} \\ P_3 \\ P_{21,22} \\ P_{22} \\ 0 \\ P_{24} \end{bmatrix}, \quad \mathbf{S}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1.00 & 1.00 & 0 \\ -2.00 & 0 & -1.00 \\ 0.48 & 0 & 0.48 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.48 & 0 & -0.48 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{V}^T = [\, P_8 \quad P_{7,8} \quad P_{8,9} \,].$$

Matrix $\mathbf{S}$ contains the linear combinations of the redundant measurements (given in $\mathbf{V}$) in terms of the required ones (given in $\mathbf{U}$). Most of the measurements are critical (those corresponding with the null columns of matrix $\mathbf{S}$). Any measurement contained in $\mathbf{U}$ can be replaced by a measurement contained in $\mathbf{V}$ if the corresponding element in $\mathbf{S}$ is nonnull. For example, active power injection P_7 can be replaced by either the active power injection P_8 or the active power flow $P_{7,8}$ but not by the active power flow $P_{8,9}$.

B. IEEE 300-Bus System

The algorithm has been tested with different combinations of measurements using the IEEE 300-bus system [22]. In all cases, the method works successfully.

At each iteration of the algorithm, different pivoting strategies have been used to avoid zero pivots (partial and complete techniques [23]) and pivot values less than 0.0001 has been considered zero. No computational difficulties pertaining to zero identification were encountered.

VI. CONCLUSION

A new algorithm that allows checking observability, determining measurement replacements, identifying critical measurements, determining observable islands, identifying irrelevant boundary injections, and selecting measurements to restore observability is presented. In addition, updating the results if new measurements become available or current measurements are lost is possible with a reduced computational overload. An example and two case studies are used to illustrate the characteristics of the algorithm proposed.

APPENDIX
PARTITIONED MATRIX ALGORITHM

The algorithm described in the previous sections can be extended to partitioned matrices as follows [2]. Consider matrix $\mathbf{H}$ partitioned as

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} & \cdots & \mathbf{H}_{1s} \\ \mathbf{H}_{21} & \mathbf{H}_{22} & \cdots & \mathbf{H}_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{H}_{r1} & \mathbf{H}_{r2} & \cdots & \mathbf{H}_{rs} \end{bmatrix} \tag{3}$$

where $\mathbf{H}_{ij}, i = 1, \ldots, r, j = 1, \ldots, s$ are submatrices of matrix $\mathbf{H}$ not necessarily squares. Note that submatrices in the same block matrix row of $\mathbf{H}$ have equal number of rows, and submatrices in the same block matrix column of $\mathbf{H}$ have identical number of columns.

Step 1) Set matrix

$$\mathbf{W} = \begin{bmatrix} \mathbf{I}_{11} & \mathbf{0}_{12} & \cdots & \mathbf{0}_{1s} \\ \mathbf{0}_{21} & \mathbf{I}_{22} & \cdots & \mathbf{0}_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{s1} & \mathbf{0}_{s2} & \cdots & \mathbf{I}_{ss} \end{bmatrix} \tag{4}$$

where $\mathbf{I}_{jj}, j = 1, \ldots, s$ is the identity matrix of dimension the number of columns of submatrix $\mathbf{H}_{ii}$, and matrix $\mathbf{0}_{ij}$, $i = 1, \ldots, s$, $j = 1, \ldots, r$ is a matrix of zeroes to complete $\mathbf{W}$. Note that $\mathbf{W}$ is a square matrix of size the number of columns of $\mathbf{H}$. Set vector $\mathbf{U} = \mathbf{0}_s$ (null vector of dimension s), and initialize vector $\mathbf{V}$ and matrix $\mathbf{S}$ to the empty sets. Let $i = 1$.

Step 2) Calculate the matrix products $\mathbf{T}_j = \mathbf{H}_i^T \mathbf{W}_j$; $j = 1, \ldots, s$, that is, the dot products of the block matrix row $\mathbf{H}_i$ by the block matrix column $\mathbf{W}_j$.

Step 3) Locate the first invertible matrix $\mathbf{T}_p$, $1 \leq p \leq s$, corresponding to a null element in position p on vector $\mathbf{U}$, which indicates the block matrix column p of matrix $\mathbf{W}$ to pivot, and replace the pth component of vector $\mathbf{U}$ by i. If there is no such a column, append i to vector $\mathbf{V}$ and matrix $\mathbf{T}$ to matrix $\mathbf{S}$ and go to Step 6). Otherwise, continue with Step 4).

Step 4) Post-multiply the p block matrix column of $\mathbf{W}$ by $\mathbf{T}_p^{-1}$, that is, $\mathbf{W_p} = \mathbf{W_p} T_\mathbf{p}^{-1}$.

Step 5) For $j = 1$ to s, $j \neq p$ and $\mathbf{T}_j \neq \mathbf{0}$ do $\mathbf{W_{kj}} = \mathbf{W_{kj}} - \mathbf{W_{kp}} \mathbf{T_j}$ for $k = 1, \ldots, s$.

Step 6) If $i = r$, continue with Step 7). Otherwise, increase i in one unit and continue with Step 2).

Step 7) If $\mathbf{U}$ only contains nonzero elements, the state is observable; otherwise, it is not. Return vectors $\mathbf{U}$ and $\mathbf{V}$ and matrices $\mathbf{S}$ and $\mathbf{W}$. Note that in this algorithm, indexes in vectors $\mathbf{U}$ and $\mathbf{V}$ correspond with the block matrix rows or columns of $\mathbf{H}$.

In particular, consider the case of the matrix $\mathbf{H}$ partitioned as in [24]

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_f & \mathbf{H}_n \\ \mathbf{H}_m & \mathbf{H}_r \end{bmatrix} \tag{5}$$

where

$\mathbf{H}_f$ square and full submatrix with dimension equal to the rank of $\mathbf{H}$;

$\mathbf{H}_m$ matrix containing rows corresponding to redundant measurements;

$\mathbf{H}_n$, $\mathbf{H}_r$ submatrices to complete the partition of $\mathbf{H}$.

Applying the algorithm above, the final vector $\mathbf{U}$ is [1, 0], the final vector $\mathbf{V}$ is [2], the final matrices $\mathbf{W}$ and $\mathbf{N}$ are

$$\mathbf{W} = \begin{bmatrix} \mathbf{H}_f^{-1} & -\mathbf{H}_f^{-1}\mathbf{H}_n \\ \mathbf{0}_{21} & \mathbf{I}_{22} \end{bmatrix}, \ \mathbf{N} = \begin{bmatrix} -\mathbf{H}_f^{-1}\mathbf{H}_n \\ \mathbf{I}_{22} \end{bmatrix} \tag{6}$$

and the final matrix $\mathbf{S}$ is $\mathbf{H}_m \mathbf{H}_f^{-1}$.

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